KWANTLEN POLYTECHNIC UNIVERSITY

Mathematics Problem of the Week

Problem 287 Solution

Cats Sharing Fish

Five cats are trapped on an island in the middle of the Atlantic Ocean. On their first day, they caught as many fish as they could and put them into a big pile. Because it is late at night, they want to divide the fish the next day. At night, each cat took a turn watching the pile while the rest slept.

- The first watcher got bored so she decided to divide the fish into 5 equal piles. When she did this, she found that she had two remaining fish. She gave the remaining two fish to a dolphin who lives nearby. Then she hid her share of fish and combined the remaining four piles back into one pile.
- The second watcher did the same thing. He divided the pile into 5 equal piles. Then, he found that he had two remaining fish. He gave the remaining two fish to the dolphin, hid his share of fish and combined the other four piles.
- The third, fourth, and fifth cat did the exact same thing. The next morning, all five cats woke up. They divided the remaining fish into 5 equal piles and each of them got one pile. This time there is no fish left.

What is the smallest number of fish there could have been in the original pile?[1]

[1]For immediate assistance with this problem, email Asiyeh at `asiyeh.sanaei@kpu.ca` or Vicky at `siqi.wei@kpu.ca`.

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Solution:

- Let N be the number of fishes cats caught.
- Let N_i, with $i \in \{1, 2, 3, 4, 5\}$, be the amount of fish left after the i-th watcher hid its own fishes and combined the remaining four piles into one.
- Let C be the number of fishes each cat get during the last dividing in the morning. (No include those they hid for themselves at night.)

Note that all $N_i - 2$, for $i \neq 5$, are divisible by 5. $N_5 = 5C$. Therefore, we have the following equations:

1. $N_1 = \frac{4}{5}(N - 2)$
2. $N_2 = \frac{4}{5}(N_1 - 2) = \frac{4}{5}(\frac{4}{5}(N - 2) - 2)$
3. $N_3 = \frac{4}{5}(N_2 - 2) = \frac{4}{5}(\frac{4}{5}(\frac{4}{5}(N - 2) - 2) - 2)$
4. $N_4 = \frac{4}{5}(N_3 - 2) = \frac{4}{5}(\frac{4}{5}(\frac{4}{5}(\frac{4}{5}(N - 2) - 2) - 2) - 2)$
5. $N_5 = \frac{4}{5}(N_4 - 2) = \frac{4}{5}(\frac{4}{5}(\frac{4}{5}(\frac{4}{5}(\frac{4}{5}(N - 2) - 2) - 2) - 2) - 2)$
6. $C = \frac{1}{5}(\frac{4}{5}(\frac{4}{5}(\frac{4}{5}(\frac{4}{5}(\frac{4}{5}(N - 2) - 2) - 2) - 2) - 2) - 2))$

Therefore:

$$C = \frac{4^5}{5^6}N - 2 \cdot \frac{4^5}{5^6} - 2 \cdot \frac{4^4}{5^5} - 2 \cdot \frac{4^3}{5^4} - 2 \cdot \frac{4^2}{5^3} - 2 \cdot \frac{4}{5^2}$$

Apply formula for sum of Geometric Series:

$$C = \frac{4^5}{5^6}N - \frac{\frac{8}{5^2}(1 - \frac{4^5}{5^5})}{1 - \frac{4}{5}}$$

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$$C = \frac{4^5}{5^6}N - \frac{8}{5}(1 - \frac{4^5}{5^5}) = \frac{4^5}{5^6}N - \frac{8}{5} + 8 \cdot \frac{4^5}{5^6}$$

$$5^5(5C + 8) = 4^5N - 8 \cdot 5^5 + 8 \cdot 4^5$$

$$5^5(5C + 8) = 4^5(N + 8)$$

$N + 8$ must be divisible by 5^5. Let $N + 8 = 5^5k$. Hence $5C + 8 = 4^5k$.
If $k = 1$, $5C = 1016$, then C is not a positive integer.
When $k = 2$, $5C = 2040 \Rightarrow C = 408$.
Therefore $k = 2$, $N + 8 = 2 \cdot 5^5 \Rightarrow N = 2 \cdot 5^5 - 8 = 6242$.

Unfortunately, we did not have any winners.