Chapter 8
Periodic Functions
8.1 Graphs of the Sine and Cosine Functions

Section Exercises
Verbal

1. Why are the sine and cosine functions called periodic functions?
   The sine and cosine functions have the property that $f(x+P)=f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis.
2. -
3. For the equation $A\cos(Bx+C)+D,$ what constants affect the range of the function and how do they affect the range?
   The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.
4. -
5. How can the unit circle be used to construct the graph of $f(t)=\sin t$?
   At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the y-coordinate of the point.

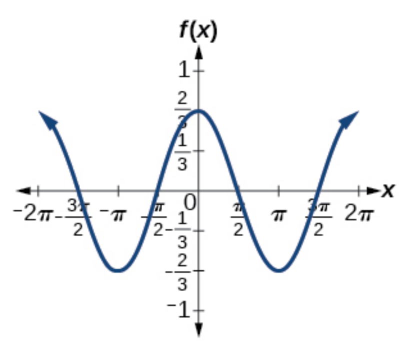
Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. Round answers to two decimal places if necessary.

6. -
7. $f(x)=\frac{2}{3}\cos x$

   $A=\frac{2}{3},\ B=1,\ C=0,\ D=0$. The amplitude is $\left|\frac{2}{3}\right|=\frac{2}{3}$, so the function is compressed. The period is $\frac{2\pi}{1}=2\pi$ so one full cycle is graphed between 0 and 2π . The midline is $y=0$.

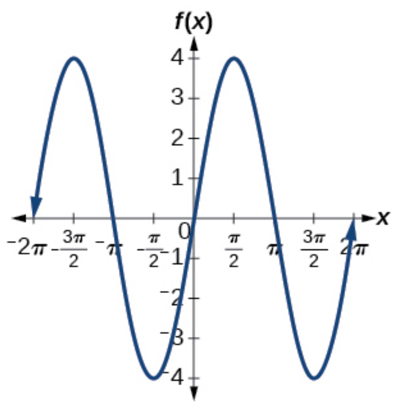
   The maximum, $y=\frac{2}{3}$, occurs at $x=2\pi$ and the minimum, $y=-\frac{2}{3}$, occurs at $x=\pi$.



8. -

9. $f(x) = 4\sin x$

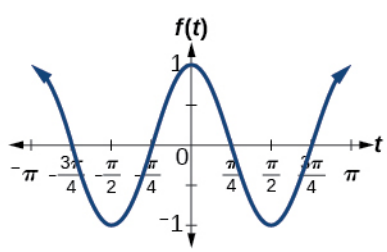
$A = 4,\ B = 1,\ C = 0,\ D = 0$. The amplitude is $|4| = 4$, so the function is stretched. The period is $\frac{2\pi}{1} = 2\pi$ so one full cycle is graphed between 0 and 2π. The midline is $y = 0$. The maximum, $y = 4$, occurs at $x = \frac{\pi}{2}$ and the minimum, $y = -4$, occurs at $x = \frac{3\pi}{2}$.



10. -

11. $f(x) = \cos(2x)$

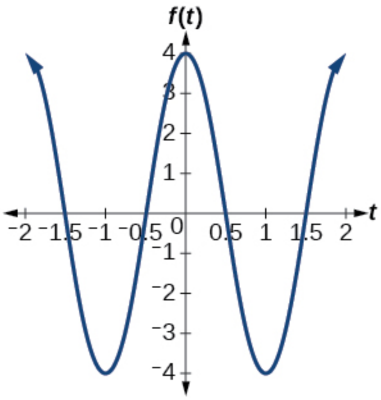
$A = 1,\ B = 2,\ C = 0,\ D = 0$. The amplitude is 1. The period is $\frac{2\pi}{2} = \pi$ so one full cycle is graphed between 0 and π. The midline is $y = 0$. The maximum, $y = 1$, occurs at $x = \pi$ and the minimum, $y = -1$ occurs at $x = \frac{\pi}{2}$.



12. -

13. $f(x)=4\cos(\pi x)$

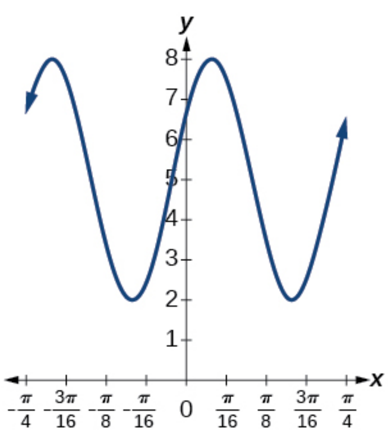
$A=4,\ B=\pi,\ C=0,\ D=0$. The amplitude is $|4|=4$, so the function is stretched. The period is $\frac{2\pi}{\pi}=2$, so one full cycle is graphed between 0 and π. The midline is $y=0$. The maximum, $y=4$, occurs at $x=2$ and the minimum, $y=-4$, occurs at $x=1$.



14. -

15. $y=3\sin(8(x+4))+5$

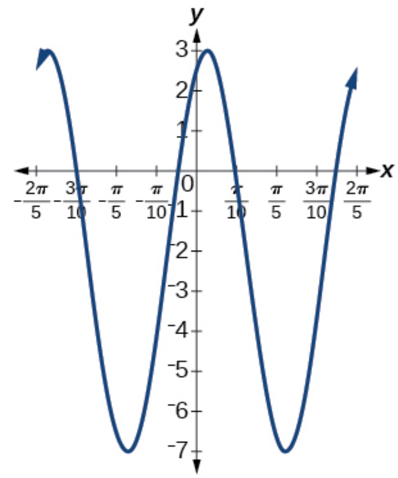
$A=3,\ B=8,\ C=-32,\ D=5$. The amplitude is $|3|=3$, so the function is stretched. The period is $\frac{2\pi}{8}=\frac{\pi}{4}$, so one full cycle is graphed between 0 and $\frac{\pi}{4}$. The midline is $y=5$. There is a horizontal shift of 4 units to the left. The maximum, $y=8$, occurs at $x=0.12$ and the minimum, $y=2$, occurs at $x=0.516$.



16. -

17. $y = 5\sin(5x+20)-2$

$A=5,\ B=5,\ C=-20,\ D=-2$. The amplitude is $|5|=5$, so the function is stretched. The period is $\frac{2\pi}{5}$, so one full cycle is graphed between 0 and $\frac{2\pi}{5}$. The midline is $y=-2$. There is a horizontal shift of 4 units to the left. The maximum, $y=3$, occurs at $x=0.08$ and the minimum, $y=-7$, occurs at $x=0.71$.

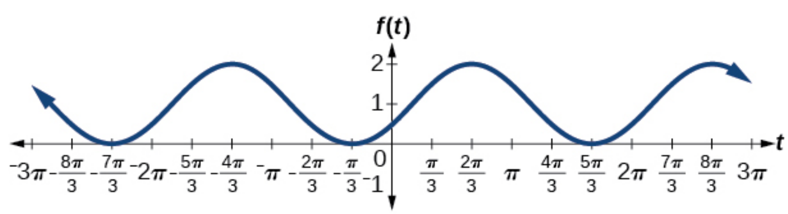


For the following exercises, graph one full period of each function, starting at $x=0$. For each function, state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. -

19. $f(t) = -\cos\left(t+\frac{\pi}{3}\right)+1$

$A=-1,\ B=1,\ C=-\frac{\pi}{3},\ D=1$. The amplitude is $|-1|=1$, so the graph is not stretched, but it is reflected across the x-axis. The period is $\frac{2\pi}{1}=2\pi$, so one full cycle is graphed between 0 and 2π. The midline is $y=1$. There is a horizontal shift of $\frac{\pi}{3}$ units to the left. The maximum, $y=2$, occurs at $t=2.09$ and the minimum, $y=0$, occurs at $t=5.24$.

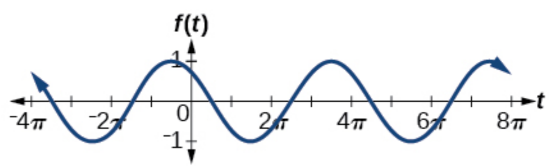


20. -

21. $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

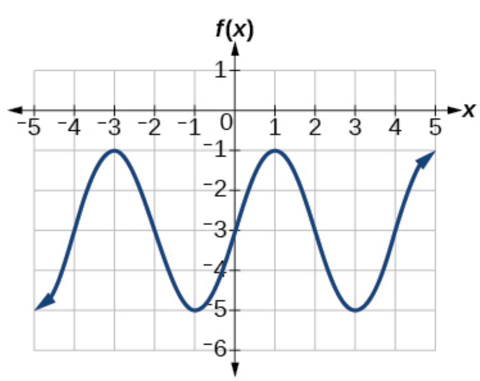
$A = -1,\ B = \frac{1}{2},\ C = -\frac{5\pi}{3},\ D = 0.$ The amplitude is $|-1| = 1$, so the graph is reflected across the x-axis. The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$, so one full cycle is graphed between 0 and 4π.

The midline is $y = 0$. There is a horizontal shift of $\frac{\frac{5\pi}{3}}{\frac{1}{2}} = \frac{10\pi}{3}$ units to the left. The maximum, $y = 1$, occurs at $t = 11.52$ and the minimum, $y = -1$, occurs at $t = 5.24$.



22. -

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in the figure.

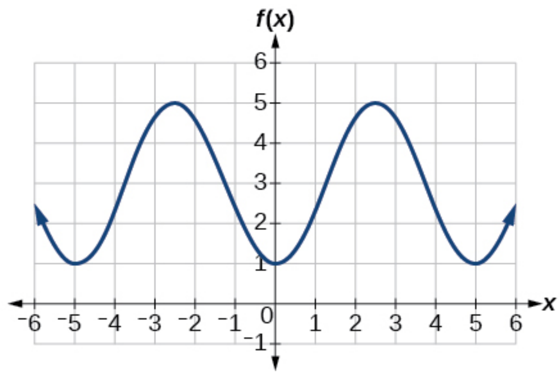


The midline is $y = -3$, so $D = -3$. The amplitude is 2, so A = 2 or –2. The function is at the midline when x = 0, so we will write this as a sine function with no horizontal shift, thus C =

0. It increases as x increases from 0, so $A = 2$. The graph goes through a full cycle in 4 units, so the period = 4. $\frac{2\pi}{B} = 4$ so $B = \frac{\pi}{2}$. This gives us the equation $f(x) = 2\sin\left(\frac{\pi}{2}x\right) - 3$.

24. -

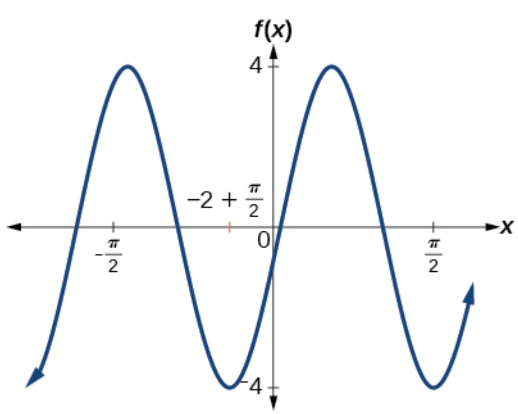
25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the figure.



The midline is $y = 3$, so $D = 3$. The amplitude is 2, so $A = 2$ or –2. The function is at its minimum when $x = 0$, so we will write this as a cosine function with $A = -2$. There is no horizontal shift, so $C = 0$. The graph goes through a full cycle in 5 units, so the period is 5. $\frac{2\pi}{B} = 5$, so $B = \frac{2\pi}{5}$. This gives us the equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

26. -

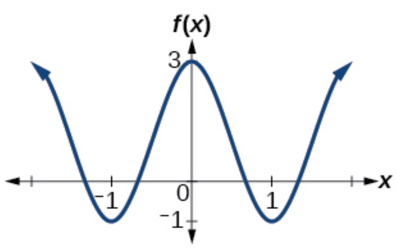
27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the figure.



The midline is $y = 0$, so $D = 0$. The amplitude is 4, so $A = 4$ or -4. The function is at its minimum when $x = -2 + \frac{\pi}{2}$. The graph goes through a full cycle in 2 units, so the period is 2. $\frac{2\pi}{B} = 2$, so $B = \pi$. We will write this as a cosine function, shifted $\frac{\pi}{2}$ units to the right. Because it is at a minimum when $x = -2 + \frac{\pi}{2}$, we will use $A = -4$. This gives us the equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$

28. -

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the figure.



The midline is $y = 1$, so $D = 1$. The amplitude is 2, so A is 2 or -2. The function is at its maximum when $x = 0$, so we will write this as a cosine function with $A = 2$. The graph goes through a full cycle in 2 units, so the period is 2. $\frac{2\pi}{B} = 2$, so $B = \pi$. There is no horizontal shift. This gives us the equation $f(x) = 2\cos(\pi x) + 1$

30. -

Algebraic

For the following exercises, let $f(x) = \sin x$.

31. On $[0, 2\pi)$, solve $f(x) = 0$.

We can determine the solution to $\sin x = 0$ by examining the unit circle. The solutions are $x = 0$ and $x = \pi$.

32. -

33. Evaluate $f\left(\frac{\pi}{2}\right)$.

We can use the unit circle to evaluate: $\sin\left(\frac{\pi}{2}\right)=1$.

34. -

35. On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

$f(x)=\sin x$ is at its maximum of 1 at $x=\frac{\pi}{2}$.

36. -

37. Show that $f(-x)=-f(x)$. This means that $f(x)=\sin x$ is an odd function and possesses symmetry with respect to ________________.

$\sin(-x)=\sin(0-x)$

$=\sin 0\cos x-\cos 0\sin x$

$0\cos x-1\sin x$

$=-\sin x$

$f(x)=\sin x$ is symmetric with respect to the origin.

For the following exercises, let $f(x)=\cos x$.

38. -

39. On $[0, 2\pi)$, solve $f(x)=\frac{1}{2}$.

We can determine the solution to $\cos x=\frac{1}{2}$ by examining the unit circle. The solutions are $x=\frac{\pi}{3}$ and $x=\frac{5\pi}{3}$.

40. -

41. On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

$f(x)=\cos x$ is at its maximum of 1 at $x=0$. $f(x)=\cos x$ is at its minimum of –1 at $x=\pi$.

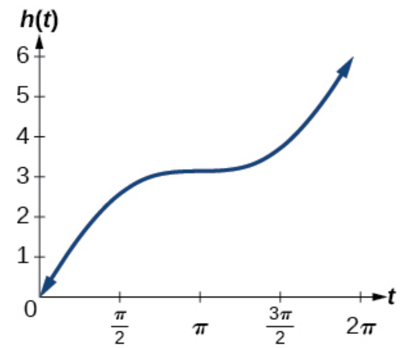
42. -

Technology

43. Graph $h(x)=x+\sin x$ on $[0,2\pi]$. Explain why the graph appears as it does.

A linear function is added to a periodic sine function. The graph does not have an amplitude because as the linear function increases without bound the combined function

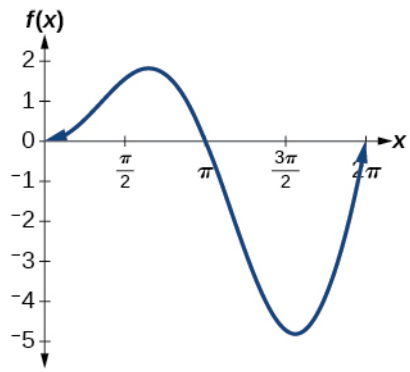
$h(x) = x + \sin x$ will increase without bound as well. The graph is bounded between the graphs of $y = x + 1$ and $y = x - 1$ because sine oscillates between -1 and 1.



44. -

45. Graph $f(x) = x\sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = \sin x$.

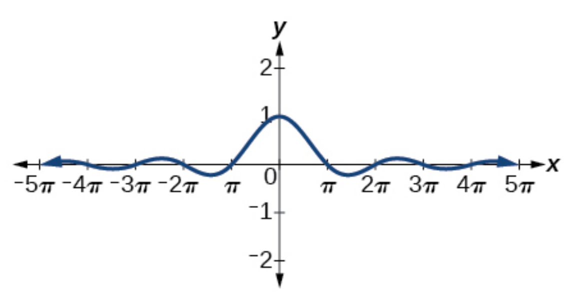
There is no amplitude because the function is not bounded.



46. -

47. Graph $f(x) = \dfrac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$ and explain what the graph shows.

The graph is symmetric with respect to the *y*-axis and there is no amplitude because the function's bounds decrease as |x| grows. There appears to be a horizontal asymptote at $y = 0$.



Real-World Applications

48. -

Chapter 8
Periodic Functions
8.2 Graphs of the Other Trigonometric Functions

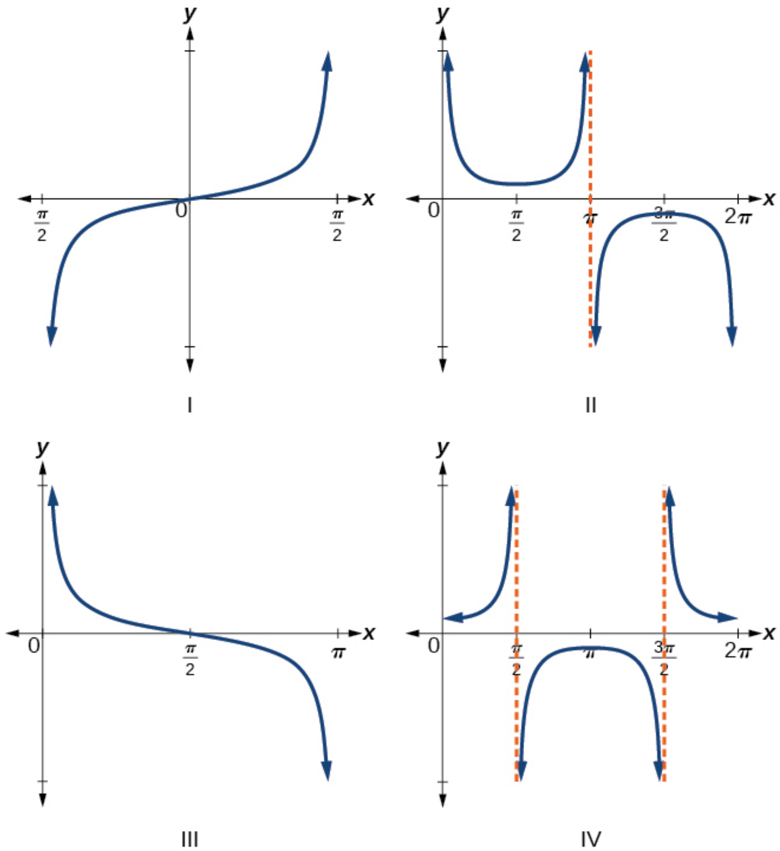
Section Exercises
Verbal

1. Explain how the graph of the sine function can be used to graph $y = \csc x$.
   Since $y = \csc x$ is the reciprocal function of $y = \sin x$, you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ to obtain the y-coordinates of $y = \csc x$. The x-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$.
2. -

3. Explain why the period of $\tan x$ is equal to π.
   Answers will vary. Using the unit circle, one can show that $\tan(x + \pi) = \tan x$.
4. -

5. How does the period of $y = \csc x$ compare with the period of $y = \sin x$?
   The period is the same: 2π.

Algebraic
For the following exercises, match each trigonometric function with one of the graphs.



6. -

7. $f\left(x\right) = \sec x$

   $\cos\left(\dfrac{\pi}{2}\right) = 0$, and $\sec x = \dfrac{1}{\cos x}$, so we are looking for a graph with a vertical asymptote $x = \dfrac{\pi}{2}$. This matches graph IV.

8. -

9. $f\left(x\right) = \cot x$

   $\tan(0) = 0$, and $\cot x = \dfrac{1}{\tan x}$, so we are looking for a graph with a vertical asymptote $x = 0$. This matches graph III.

For the following exercises, find the period and horizontal shift of each of the functions.

10. -

11. $h(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$

$A = 2,\ B = \frac{\pi}{4},\ C = -\frac{\pi}{4}$ and $D = 0.$ The graph is stretched vertically by a factor of 2. The period is $\frac{2\pi}{\frac{\pi}{4}} = 8$, and the graph has a horizontal shift of $\frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1$ unit to the left.

12. -

13. If $\tan x = -1.5,$ find $\tan(-x).$

The tangent function is odd, so $\tan(-x) = -\tan x = -(-1.5) = 1.5.$

14. -

15. If $\csc x = -5,$ find $\csc(-x).$

The cosecant function is odd, so $\csc(-x) = -\csc x = -(-5) = 5$.

16. -

For the following exercises, rewrite each expression such that the argument x is positive.

17. $\cot(-x)\cos(-x) + \sin(-x)$

$\cot(-x)\cos(-x) + \sin(-x) = -\cot x \cos x - \sin x.$

18. -

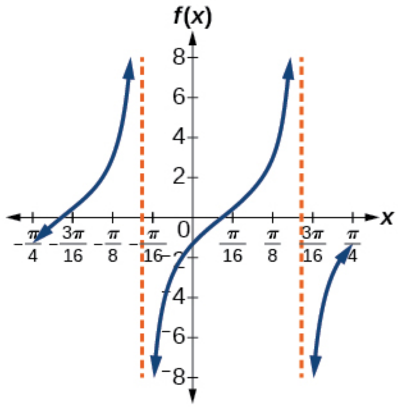
Graphical

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19. $f(x) = 2\tan(4x - 32)$

$A = 2,\ B = 4,\ C = 32$ and $D = 0.$ The stretching factor is 2. The period is $\frac{\pi}{4}.$ There is a horizontal shift of $\frac{32}{4} = 8$ units to the right. The vertical asymptotes occur at

$x = \left(\frac{\pi}{8} + \frac{\pi k}{4}\right) + 8 = \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8$, where k is an integer.



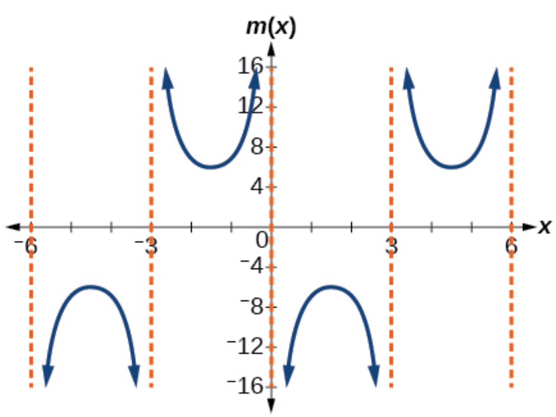
stretching factor: 2; period: $\frac{\pi}{4}$; asymptotes: $x = \frac{1}{4}\left(\frac{\pi}{2} + k\pi\right) + 8$

20. -

21. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

$A = 6,\ B = \frac{\pi}{3},\ C = -\pi$ and $D = 0$. The stretching factor is 6. The period is $\frac{2\pi}{\frac{\pi}{3}} = 6$.

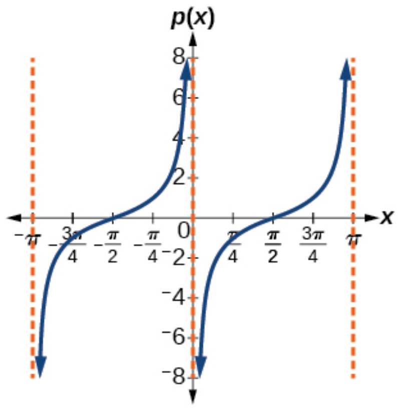
There is a horizontal shift of $\frac{-\pi}{\frac{\pi}{3}} = -3;\ 3$ units to the left. The vertical asymptotes occur at $x = \frac{\pi}{\frac{\pi}{3}}k = 3k$, where k is an integer.



22. -

23. $p(x) = \tan\left(x - \frac{\pi}{2}\right)$

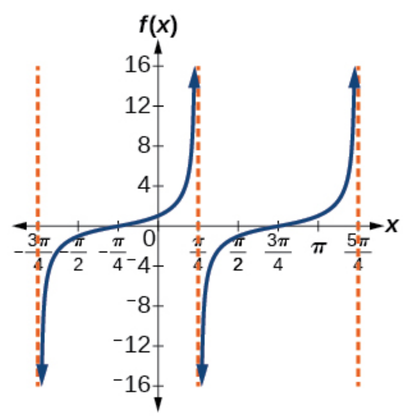
$A = 1,\ B = 1,\ C = \frac{\pi}{2}$ and $D = 0$. The stretching factor is 1. The period is $\frac{\pi}{1} = \pi$. There is a horizontal shift of $\frac{\frac{\pi}{2}}{1} = \frac{\pi}{2}$ units to the right. The vertical asymptotes occur at $x = \left(\frac{\pi}{2} + \frac{k\pi}{1}\right) + \frac{\pi}{2} = \pi + \pi k = k\pi$, where k is an integer.



24. -

25. $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

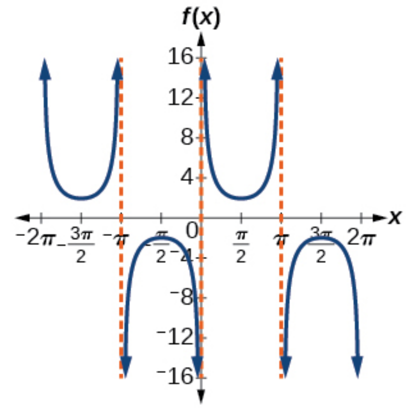
$A = 1,\ B = 1,\ C = -\frac{\pi}{4}$ and $D = 0$. The stretching factor is 1. The period is $\frac{\pi}{1} = \pi$. There is a horizontal shift of $\frac{-\frac{\pi}{4}}{1} = -\frac{\pi}{4};\ \frac{\pi}{4}$ units to the left. The vertical asymptotes occur at $x = \frac{\pi}{2} + \pi k - \frac{\pi}{4} = \frac{\pi}{4} + \pi k$, where k is an integer.



26. -

27. $f(x) = 2\csc(x)$

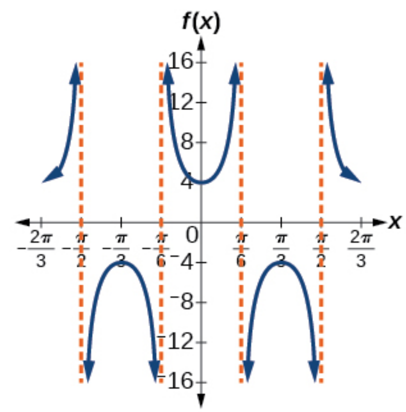
$A = 2$, $B = 1$, $C = 0$ and $D = 0$. The stretching factor is 2. The period is $\frac{2\pi}{1} = 2\pi$. There is no horizontal shift. The asymptotes occur at $x = \pi k$, where k is an integer.



28. -

29. $f(x) = 4\sec(3x)$

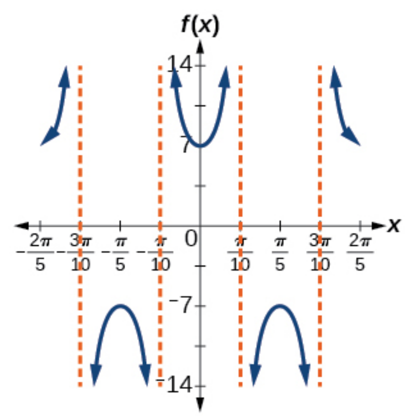
$A = 4$, $B = 3$, $C = 0$ and $D = 0$. The stretching factor is 4. The period is $\frac{2\pi}{3}$. There is no horizontal shift. The vertical asymptotes occur at $x = \frac{\pi}{2 \cdot 3}k = \frac{\pi}{6}k$, where k is an odd integer.



30. -

31. $f(x) = 7\sec(5x)$

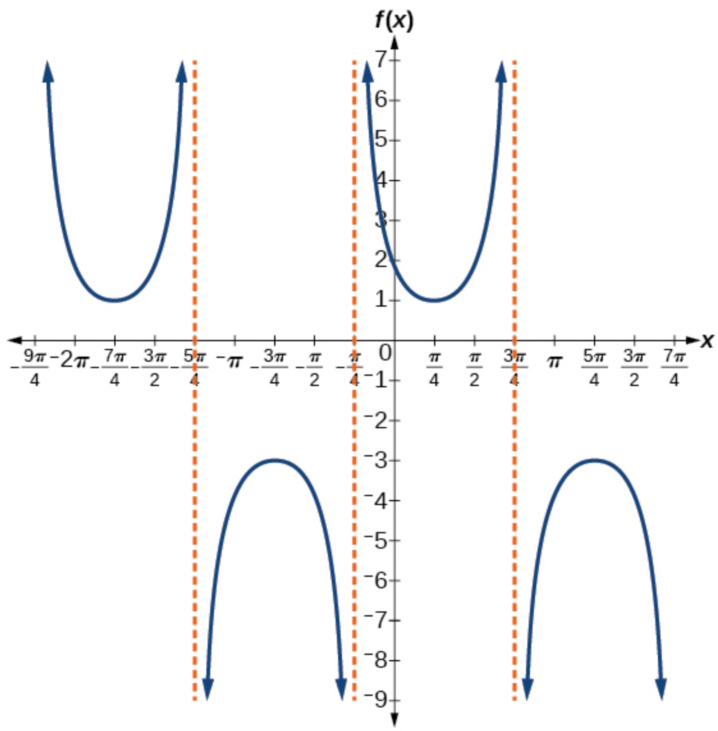
$A = 7$, $B = 5$, $C = 0$ and $D = 0$. The stretching factor is 7. The period is $\frac{2\pi}{5}$. There is no horizontal shift. The asymptotes occur at $x = \frac{\pi}{2 \cdot 5}k = \frac{\pi}{10}k$, where k is an odd integer.



32. -

33. $f(x)=2\csc\left(x+\frac{\pi}{4}\right)-1$

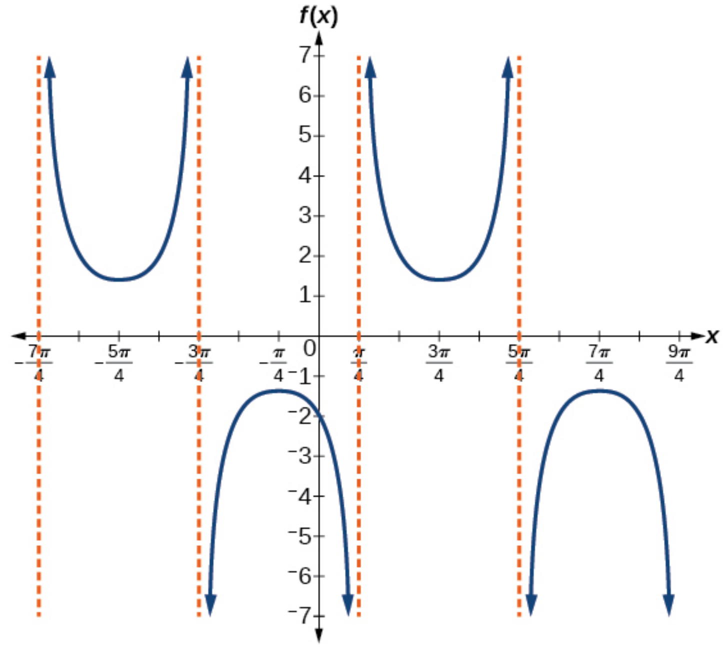
$A=2,\ B=1,\ C=-\frac{\pi}{4}$ and $D=-1$. The stretching factor is 2. The period is $\frac{2\pi}{1}=2\pi$. There is a horizontal shift of $-\frac{\pi}{4}$; $\frac{\pi}{4}$ units to the left. There is a vertical shift of 1 unit down. The vertical asymptotes occur at $x=-\frac{\pi}{4}+\frac{\pi}{1}k=-\frac{\pi}{4}+\pi k$, where k is an integer.



stretching factor: 2; period: 2π; asymptotes: $x=-\frac{\pi}{4}+\pi k$, where k is an integer.

34. -

35. $f(x)=\frac{7}{5}\csc\left(x-\frac{\pi}{4}\right)$



$A=\frac{7}{5}$, $B=1$, $C=\frac{\pi}{4}$ and $D=0$.
The stretching factor is $\frac{7}{5}$.

The period is $\frac{2\pi}{1}=2\pi$.

There is a horizontal shift of $\frac{\pi}{4}$ units to the right. The vertical asymptotes occur at $x=\frac{\pi}{1}k+\frac{\pi}{4}=\frac{\pi}{4}+k\pi$, where k is an integer.

36. -

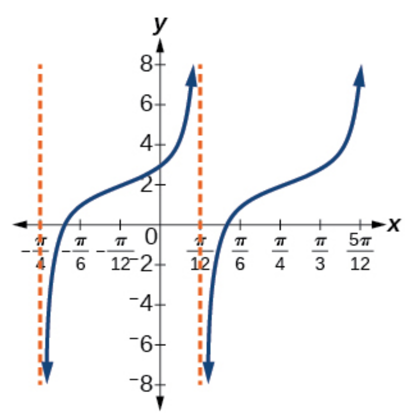
For the following exercises, find and graph two periods of the periodic function with the given stretching factor, $|A|$, period, and phase shift.

37. A tangent curve, $A=1$, period of $\frac{\pi}{3}$; and shift $(h,k)=\left(\frac{\pi}{4},2\right)$.

From the information given, we have $A=1$. The period is $\frac{\pi}{3}$, and $\frac{\pi}{B}=\frac{\pi}{3}$, so $B=3$. There is a horizontal shift of $\frac{\pi}{4}$, and $\frac{C}{B}=\frac{C}{3}=\frac{\pi}{4}$, which gives us $C=\frac{3\pi}{4}$. There is a vertical shift of 2, so $D=2$.

This gives us the equation:

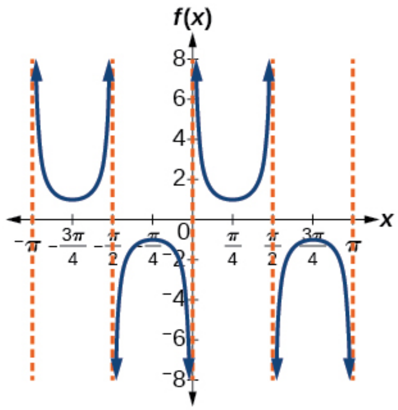
$$y=\tan\left(3x-\frac{3\pi}{4}\right)+2=\tan\left(3\left(x-\frac{\pi}{4}\right)\right)+2.$$



38. -

For the following exercises, find an equation for the graph of each function.

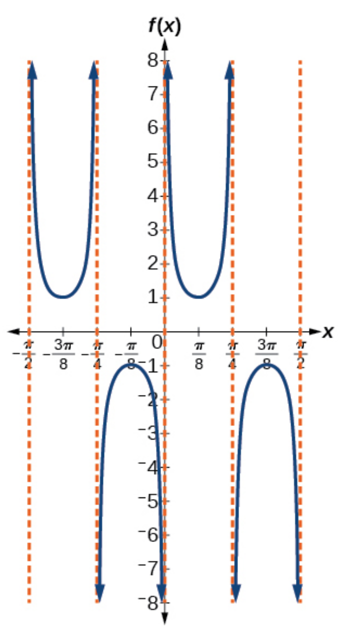
39.



This function has a vertical asymptote at $x = 0$, so we will write it as a cosecant function with $C = 0$. The minimum of the positive branches is 1, so $|A| = 1$. The function is positive immediately to the right of x = 0, so $A = 1$. The period is π, and $\frac{2\pi}{B} = \pi$, so $B = 2$. The midline is $y = 0$, so $D = 0$. This gives us the equation $f(x) = \csc(2x)$.

40. -

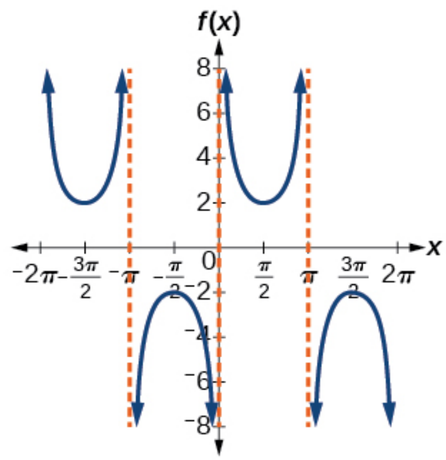
41.



This graph has a minimum at $x=\frac{\pi}{8}$, and a vertical asymptote at $x=0$, so we will write this as a cosecant function with $C=0$. The minimum of the positive branches is 1, so $|A|=1$. The graph is positive immediately to the right of $x = 0$, so $A=1$. The period is $\frac{\pi}{2}$, and $\frac{2\pi}{B}=\frac{\pi}{2}$, so $B=4$. The midline is $y=0$, so $D=0$. This gives us the equation $f(x)=\csc(4x)$.

42. -

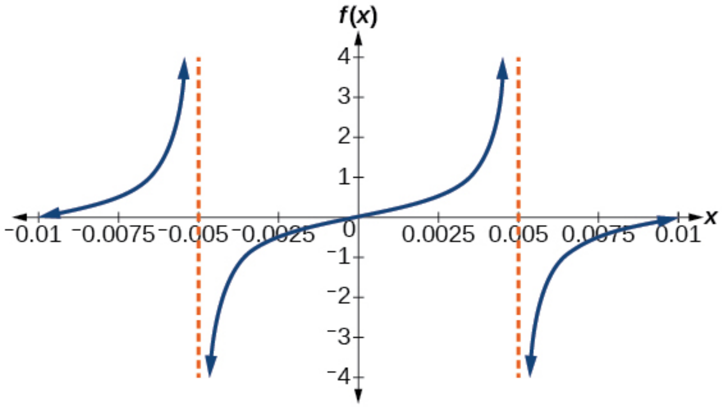
43.



This graph has a vertical asymptote at $x=0$, so we will write it as a cosecant function with $C=0$. The minimum of the positive branches is 2, so $|A|=2$. The function is positive immediately to the right of $x = 0$, so $A=2$. The period is 2π, so $B=1$. The midline is $y=0$, so $D=0$. This gives us the equation $f(x)=2\csc x$.

44. -

45.



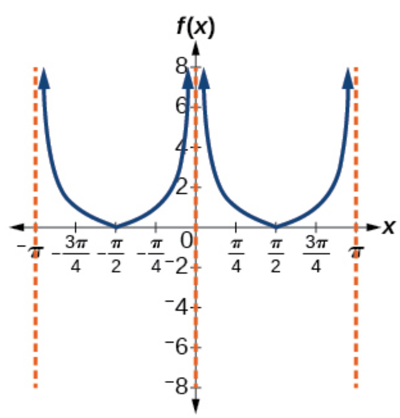
This graph is increasing with an x-intercept at $x = 0$, so we will write this as a tangent function with $C = 0$. The period is 0.01, and $\frac{\pi}{B} = 0.01$, so $B = 100\pi$. It appears that $f(0.0025) = \frac{1}{2}$, so the stretching factor is $\frac{1}{2}$. The midline is $y = 0$, so $D = 0$. This gives us the equation $f(x) = \frac{1}{2}\tan(100\pi x)$

Technology

For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$.

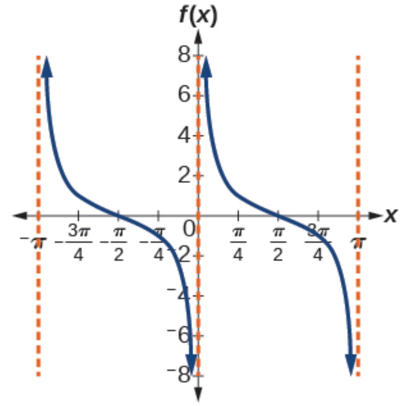
46. -

47. $f(x) = \left|\cot\left(x\right)\right|$



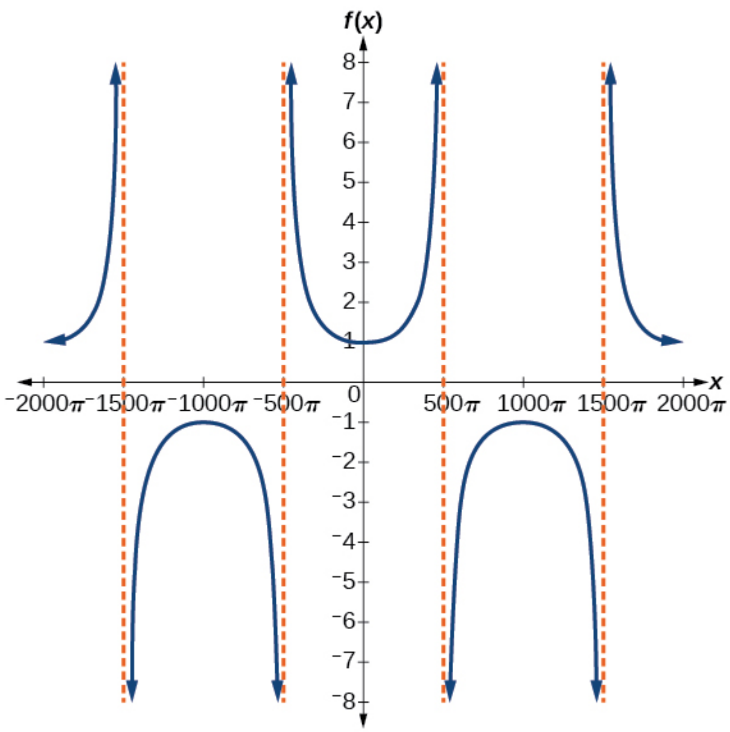
48. -

49. $f(x)=\dfrac{\csc(x)}{\sec(x)}$



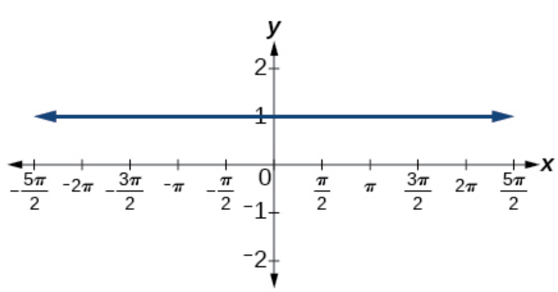
50. -

51. $f(x)=\sec(0.001x)$



52. -

53. $f(x)=\sin^2 x+\cos^2 x$

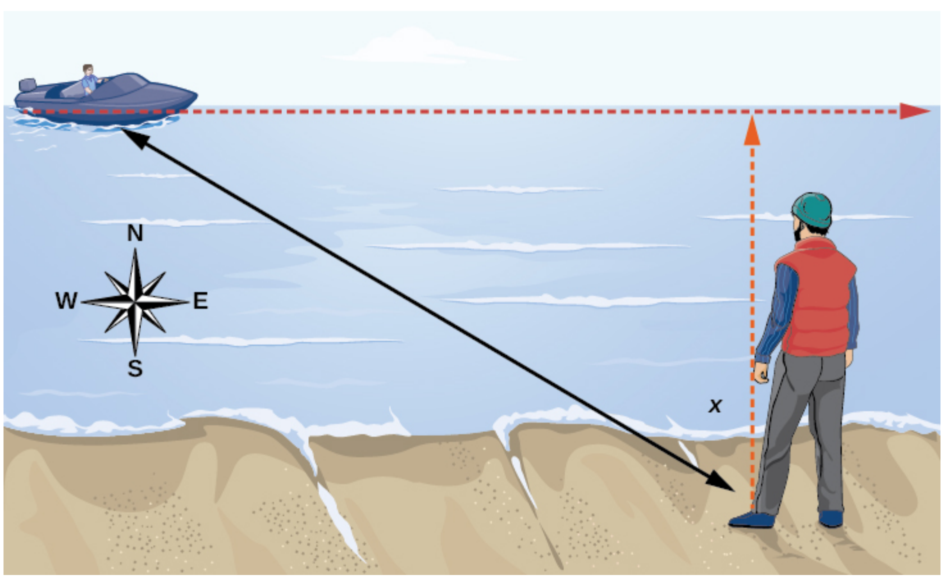


Real-World Applications

54. -

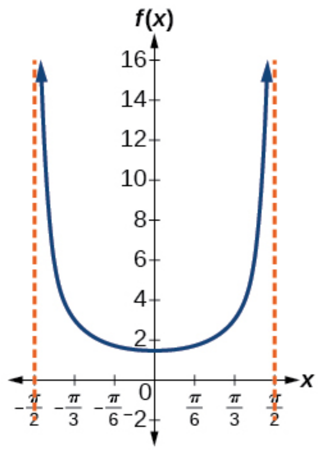
55. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See figure.)The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$.

    a. What is a reasonable domain for $d(x)$?
    b. Graph $d(x)$ on this domain.
    c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.
    d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
    e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
    f. What is the minimum distance between the fisherman and the boat? When does this occur?



a. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;

b. $A = 1.5$, $B = 1$, $C = 0$ and $D = 0$. The function will have vertical asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. It will have a minimum value of 1.5 at $x = 0$.



c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$; the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$— i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it; as $x \to \pm\frac{\pi}{2}$ this model breaks down.

d. 3; when $x = -\frac{\pi}{3}$, the boat is 3 km away;

e. 1.73; when $x = \frac{\pi}{6}$, the boat is about 1.73 km away;

f. 1.5 km; when $x = 0$

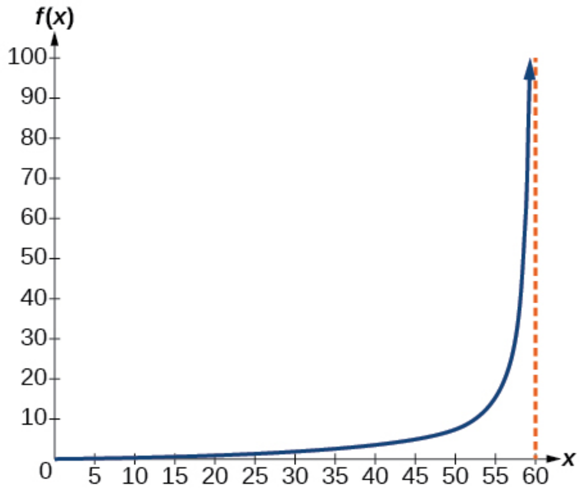
56. -

57. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

  a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
  b. Graph $h(x)$ on the interval $(0, 60)$.
  c. Evaluate and interpret the values $h(0)$ and $h(30)$.
  d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

a. $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$;

b. $A = 2$, $B = \frac{\pi}{120}$, $C = 0$ and $D = 0$. The function will have an x-intercept at $x = 0$, and a vertical asymptote at $x = 60$.



c. c. $h(0) = 0$: after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$: after 30 seconds, the rockets is 2 mi high;

d. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. As $x \to 60$ the model breaks down, since it assumes that the angle of elevation continues to increase with x. In fact, the angle is bounded at 90 degrees.

Chapter 8
Periodic Functions
8.3 Inverse Trigonometric Functions

Section Exercises
Verbal

1. Why do the functions $f(x)=\sin^{-1}x$ and $g(x)=\cos^{-1}x$ have different ranges?

   The function $y=\sin x$ is one-to-one on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y=\sin x$, $f(x)=\sin^{-1}x$. The function $y=\cos x$ is one-to-one on $[0,\pi]$; thus, this interval is the range of the inverse function of $y=\cos x$, $f(x)=\cos^{-1}x$.

2. -

3. Explain the meaning of $\frac{\pi}{6}=\arcsin(0.5)$.

   $\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

4. -

5. Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for the inverse sine function to exist?

   In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse.

6. -

7. Determine whether the following statement is true or false and explain your answer:

   $\arccos(-x)=\pi-\arccos x$.

   True . The angle, θ_1 that equals $\arccos(-x)$, $x>0$, will be a second quadrant angle with reference angle, θ_2, where θ_2 equals $\arccos x$, $x>0$. Since θ_2 is the reference angle for θ_1, $\theta_2=\pi-\theta_1$ and $\arccos(-x)=\pi-\arccos x$

Algebraic

For the following exercises, evaluate the expressions.

8. -

9. $\sin^{-1}\left(-\frac{1}{2}\right)$

We need to find the angle, x, in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin x = -\frac{1}{2}$: $x = -\frac{\pi}{6}$

10. -

11. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

We need to find the angle, x, in the interval $[0, \pi]$ such that $\cos x = -\frac{\sqrt{2}}{2}$: $x = \frac{3\pi}{4}$

12. -

13. $\tan^{-1}\left(-\sqrt{3}\right)$

We need to find the angle, x, in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\tan x = -\sqrt{3}$: $x = -\frac{\pi}{3}$

14. -

15. $\tan^{-1}\left(\sqrt{3}\right)$

We need to find the angle, x, in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\tan x = \sqrt{3}$: $x = \frac{\pi}{3}$

16. -

For the following exercises, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17. $\cos^{-1}\left(-0.4\right)$

$\cos^{-1}(-0.4) \approx 1.98$

18. -

19. $\arccos\left(\frac{3}{5}\right)$

$\arccos\left(\frac{3}{5}\right) \approx 0.93$

20. -

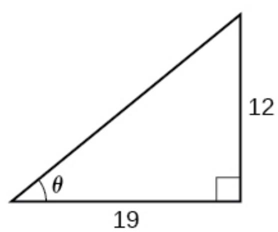
21. $\tan^{-1}(6)$

$\tan^{-1}(6) \approx 1.41$

For the following exercises, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22. -

23.



Because we know the opposite and adjacent sides, we will use the tangent function.

$\tan\theta = \frac{12}{19}$; $\tan^{-1}\left(\frac{12}{19}\right) \approx 0.56$ radian.

For the following exercises, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24. -

25. $\tan^{-1}(\sin(\pi))$

$\sin\pi = 0$, so $\tan^{-1}(\sin\pi) = \tan^{-1}(0) = 0$.

26. -

27. $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

$\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, so we can use a calculator to evaluate $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\approx 0.71$ radians

28. -

29. $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

$\sin\left(\frac{4\pi}{3}\right)=-\frac{\sqrt{3}}{2}$, so we can use a calculator to evaluate $\tan^{-1}\left(-\frac{\sqrt{3}}{2}\right)\approx -0.71$ radians

30. -

31. $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

$\sin\left(-\frac{5\pi}{2}\right)=-1$, so $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)=\tan^{-1}(-1)=-\frac{\pi}{4}$ radians

32. -

33. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

We are looking for θ such that $\cos\theta=\frac{3}{5}$. This gives us an adjacent side of 3 and a hypotenuse of 5. Using the Pythagorean Theorem, we get an opposite side of 4. So $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)=\frac{4}{5}$.

34. -

35. $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

We are looking for θ such that $\tan\theta=\frac{5}{12}$. This gives us an opposite side of 5 and an adjacent side of 12. Using the Pythagorean Theorem, we get a hypotenuse of 13. So $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)=\frac{5}{13}$.

36. -

For the following exercises, find the exact value of the expression in terms of x with the help of a reference triangle.

37. $\tan\left(\sin^{-1}(x-1)\right)$

We are looking for θ such that $\sin\theta = x-1 = \frac{x-1}{1}$. This gives us an opposite side of $x-1$ and a hypotenuse of 1. Using the Pythagorean Theorem, we get an adjacent side of $\sqrt{2x-x^2}$. So $\tan\left(\sin^{-1}(x-1)\right) = \frac{x-1}{\sqrt{-x^2+2x}}$.

38. -

39. $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

We are looking for θ such that $\sin\theta = \frac{1}{x}$. This gives us an opposite side of 1 and a hypotenuse of x. Using the Pythagorean Theorem, we get an adjacent side of $\sqrt{x^2-1}$. So $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = \frac{\sqrt{x^2-1}}{x}$.

40. -

41. $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

We are looking for θ such that $\sin\theta = x+\frac{1}{2} = \frac{2x+1}{2}$. This gives us an opposite side of $2x+1$ and a hypotenuse of 2. Using the Pythagorean Theorem, we get an adjacent side of $\sqrt{4-(2x+1)^2}$. So $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right) = \frac{2x+1}{\sqrt{4-(2x+1)^2}} = \frac{x+0.5}{\sqrt{-x^2-x+\frac{3}{4}}}$.

Extensions

For the following exercises, evaluate the expression without using a calculator. Give the exact value.

42. -

For the following exercises, find the function if $\sin t = \frac{x}{x+1}$.

43. $\cos t$

We have an opposite side of x and a hypotenuse of $x+1$. Using the Pythagorean Theorem, we get an adjacent side of $\sqrt{2x+1}$. So $\cos t = \dfrac{\sqrt{2x+1}}{x+1}$.

44. -

45. $\cot t$

From our results in number 43, we have $\cot t = \dfrac{\sqrt{2x+1}}{x}$.

46. -

47. $\tan^{-1}\left(\dfrac{x}{\sqrt{2x+1}}\right)$

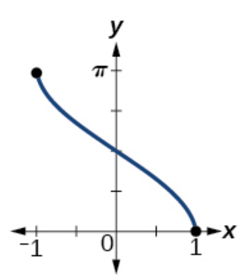
From the results in number 43, we can see that $\tan t = \dfrac{x}{\sqrt{2x+1}}$. Thus $\tan^{-1}\left(\dfrac{x}{\sqrt{2x+1}}\right) = t$.

Graphical

48. -

49. Graph $y = \arccos x$ and state the domain and range of the function.

Using the graph of $y = \cos x$ on the restricted domain $[0,\ \pi]$ we get the graph of the inverse function, $y = \arccos x$:



The domain is $[-1,\ 1]$ and the range is $[0,\ \pi]$.

50. -

51. For what value of x does $\sin x = \sin^{-1} x$? Use a graphing calculator to approximate the answer.

The graphs of $y = \sin x$ and $y = \sin^{-1} x$ intersect at $x = 0$.

52. -

Real-World Applications

53. Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

The length of the ladder is the hypotenuse, and the height that it reaches is the adjacent side. This means $\cos\theta = \frac{12}{13}$. We can use a calculator to evaluate $\cos^{-1}\left(\frac{12}{13}\right) \approx 0.395$ radian.

54. -

55. An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch. side.

We need to find θ such that the adjacent side is 4 inches and the hypotenuse is 9 inches. This means $\cos\theta = \frac{4}{9}$. We can use a calculator to evaluate $\cos^{-1}\left(\frac{4}{9}\right) \approx 1.11$ radians.

56. -

57. A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

We need to find θ such that the opposite side is 12 feet and the adjacent side is 4 feet. This means $\tan\theta = \frac{12}{4} = 3$. We can use a calculator to evaluate $\tan^{-1}(3) \approx 1.25$ radians.

58. -

59. The line $y = \frac{-3}{7}x$ passes through the origin in the *x*,*y*-plane. What is the measure of the angle that the line makes with the negative *x*-axis?

The opposite side is 3 and the adjacent side is 7, so we need to find θ such that $\tan\theta = \frac{3}{7}$. We can use a calculator to determine that $\tan^{-1}\left(\frac{3}{7}\right) \approx 0.405$ radians.

60. -

61. A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

We have a hypotenuse of 20 ft. and an adjacent side of 10 ft. We need to find the angle θ such that $\cos\theta = \frac{10}{20}$. From our unit circle, we can determine that $\theta = 60^\circ$, so this is not within the specifications.

62. -

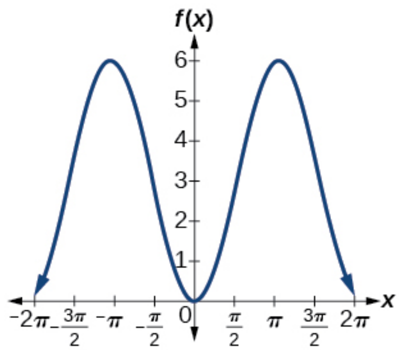
Chapter 8 Review Exercises

Section 8.1

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1. $f(x) = -3\cos x + 3$

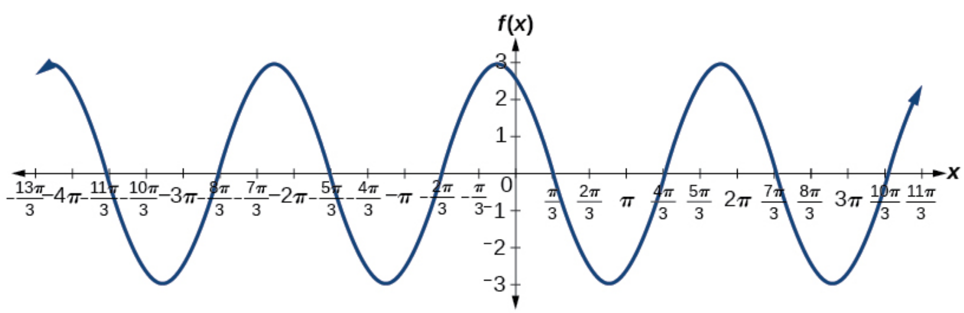
   $A = -3,\ B = 1,\ C = 0$ and $D = 3$. The amplitude is $|-3| = 3$, so the function is stretched by a factor of 3, and A is negative, so the function is reflected horizontally. The period is $\frac{2\pi}{1} = 2\pi$, so one full cycle is graphed between 0 and 2π. The midline is $y = 3$. The maximum, $y = 6$, occurs at $x = \pi$ and the minimum, $y = 0$, occurs at $x = 0$.



2. -

3. $f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

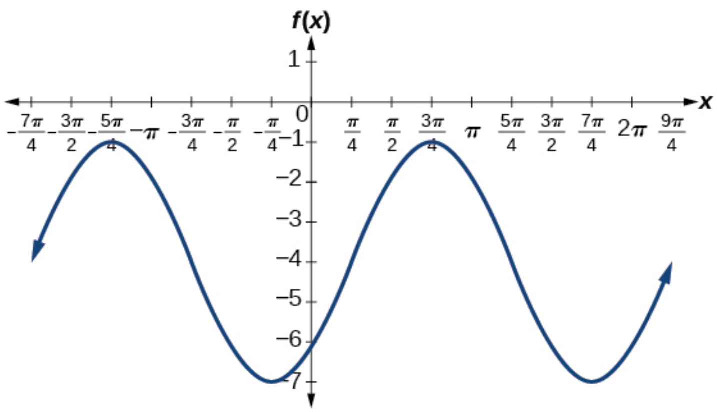
   $A = 3,\ B = 1,\ C = -\frac{\pi}{6}$ and $D = 0$. The amplitude is $|3| = 3$, so the function is stretched by a factor of 3. The period is $\frac{2\pi}{1} = 2\pi$, so one full cycle is graphed between 0 and 2π. There is a horizontal shift of $\frac{\pi}{6}$ units to the left. The midline is $y = 0$. The maximum, y = 3, occurs at $x = -\frac{\pi}{6}$ and the minimum, y = –3, occurs at $x = \frac{5\pi}{6}$.



4. -

5. $f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4$

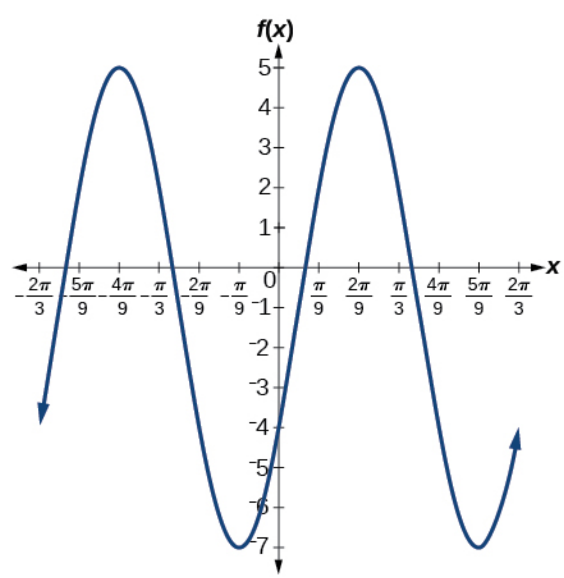
$A = 3,\ B = 1,\ C = \frac{\pi}{4}$ and $D = -4$. The amplitude is $|3| = 3$, so the function is stretched by a factor of 3. The period is $\frac{2\pi}{1} = 2\pi$, so one full cycle is graphed between 0 and 2π. There is a horizontal shift of $\frac{\pi}{4}$ units to the right. The midline is $y = -4$. The maximum, $y = -1$, occurs at $x = \frac{3\pi}{4}$ and the minimum, $y = -7$, occurs at $x = -\frac{\pi}{4}$.



6. -

7. $f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1$

$A = 6,\ B = 3,\ C = \frac{\pi}{6}$ and $D = -1$. The amplitude is $|6| = 6$, so the function is stretched by a factor of 6. The period is $\frac{2\pi}{3}$, so one full cycle is graphed between 0 and $\frac{2\pi}{3}$. There is a horizontal shift of $\frac{\frac{\pi}{6}}{3} = \frac{\pi}{18}$ units to the right. The midline is $y = -1$. The maximum, $y = 5$, occurs at $x = \frac{2\pi}{9}$ and the minimum, $y = -7$, occurs at $x = -\frac{\pi}{9}$.

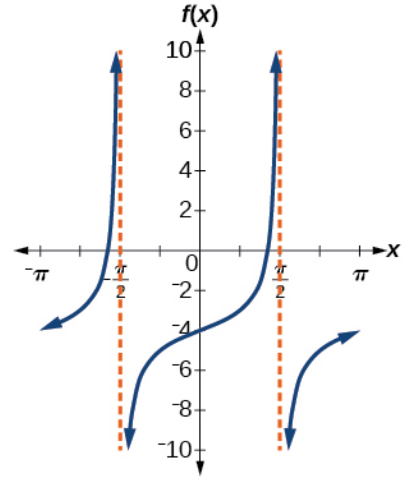


8. -

Section 8.2

9. $f(x) = \tan x - 4$

$A = 1,\ B = 1,\ C = 0$ and $D = -4$. There is no vertical stretch. The period is $\frac{\pi}{1} = \pi$. There is no horizontal shift. The midline is $y = -4$. The vertical asymptotes occur at $x = \frac{\pi}{2} + \pi k$, where k is an integer.

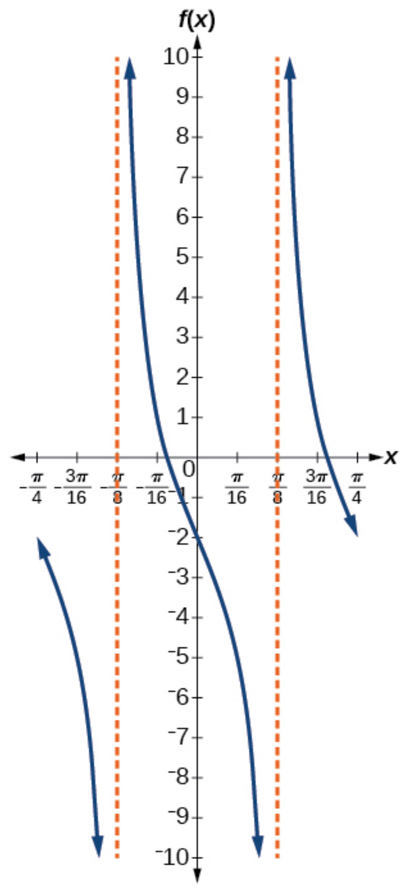


10. -

11. $f(x) = -3\tan(4x) - 2$

$A = -3,\ B = 4,\ C = 0$ and $D = -2$. The stretching factor is $|-3| = 3$. The period is $\frac{\pi}{4}$. There is no horizontal shift. The midline is $y = -2$. The vertical asymptotes occur at $\frac{\pi}{8} + \frac{\pi}{4}k$

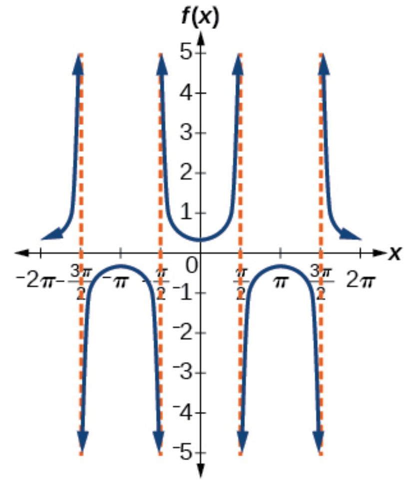
stretching factor: 2; period: $\frac{\pi}{4}$; midline: $y = -2$; asymptotes: $x = \frac{\pi}{8} \pm \frac{n\pi}{4}$



12. -

For the following exercises, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

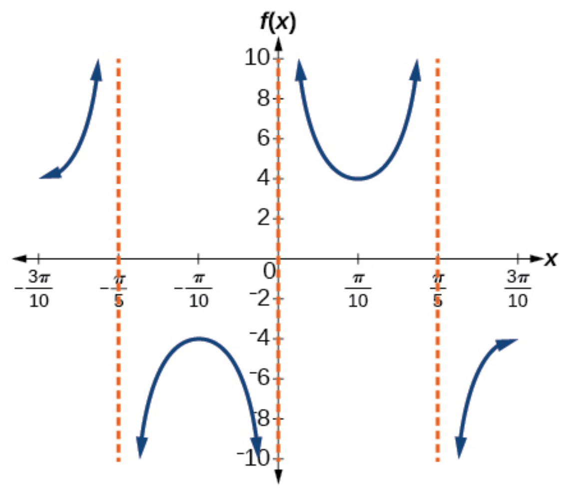
13. $f(x) = \frac{1}{3}\sec x$



$A = \frac{1}{3}$, $B = 1$, $C = 0$ and $D = 0$. The period is $\frac{2\pi}{1} = 2\pi$. There is no phase shift. The midline is $y = 0$. The function has local minimum values of $y = \frac{1}{3}$ and local maximum values of $y = -\frac{1}{3}$. The vertical asymptotes occur at $x = \frac{\pi}{2}k$, where k is an odd integer.

14. -

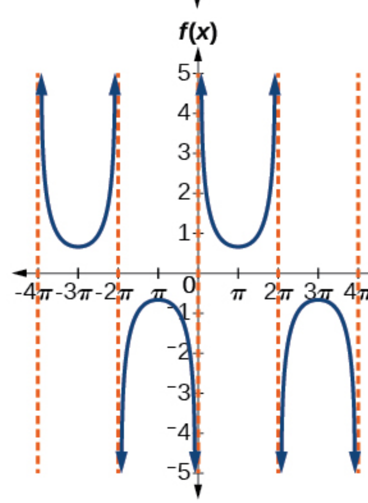
15. $f(x) = 4\csc(5x)$



$A = 4$, $B = 5$, $C = 0$ and $D = 0$. The period is $\frac{2\pi}{5}$. There is no phase shift. The midline is $y = 0$. The function has local minimum values of $y = 4$ and local maximum values of $y = -4$. The vertical asymptotes occur at $x = \frac{\pi}{5}k$, where k is an integer.

16. -

17. $f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right)$



$A = \frac{2}{3}$, $B = \frac{1}{2}$, $C = 0$ and $D = 0$. The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. There is no phase shift. The midline is at $y = 0$. The function has local minimum values of $y = \frac{2}{3}$ and local maximum values of $y = -\frac{2}{3}$. The vertical asymptotes occur at $x = \frac{\pi}{\frac{1}{2}}k = 2\pi k$, where k is an integer.

18. -

For the following exercises, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: $y = 12{,}000 + 8{,}000\sin(0.628x)$, where the domain is the years since 1980 and the range is the population of the city.

19. What is the largest and smallest population the city may have?
    The largest population the city may have is 20,000. The smallest population the city may have is 4,000.
20. -
21. What are the amplitude, period, and phase shift for the function?
    $A = 8000$, $B = 0.628$, $C = 0$ and $D = 12{,}000$. The amplitude is 8,000. The period is $\frac{2\pi}{0.628} \approx 10$. The phase shift is 0.
22. -
23. What is the predicted population in 2007? 2010?

    In 2007, the predicted population is $12{,}000 + 8{,}000\sin(0.628 \cdot 27) = 4{,}413$. In 2010, the population will be $12{,}000 + 8{,}000\sin(0.628 \cdot 30) = 11{,}924$.

For the following exercises, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

24. -
25. At time = 0, what is the displacement of the weight?
    At $x = 0$, the displacement is $5\cos\left(\frac{\pi}{5} \cdot 0\right) = 5$ inches.
26. -
27. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?
    The period is 10 seconds

Section 8.3

For the following exercises, find the exact value without the aid of a calculator.

28. -
29. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

    We need to find the angle, x, in the interval $[0, \pi]$ such that $\cos x = \frac{\sqrt{3}}{2}$: $x = \frac{\pi}{6}$
30. -

31. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

We need to find the angle, x, in the interval $[0,\ \pi]$ such that $\cos x = \frac{1}{\sqrt{2}}$: $x = \frac{\pi}{4}$

32. -

33. $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

From the unit circle we have $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, so we need to find the angle x in the interval $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ such that $\sin x = \frac{\sqrt{3}}{2}$: $x = \frac{\pi}{3}$

34. -

35. $\sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right)$

We are looking for θ such that $\sec\theta = \frac{3}{5}$. This gives us an adjacent side of 5 and a hypotenuse of 3. It is impossible for the hypotenuse to be smaller than one of the sides. This cannot be evaluated because $\frac{3}{5}$ is not in the domain of the inverse secant function. No solution.

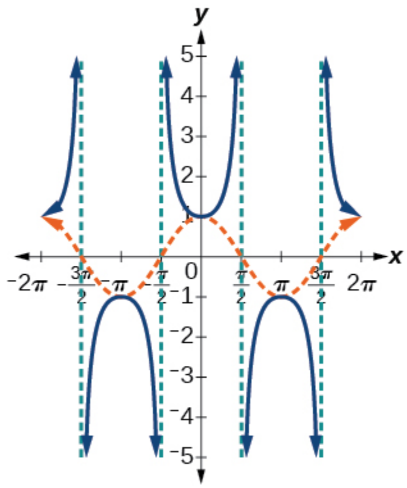
36. -

37. $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

We are looking for θ such that $\cos\theta = \frac{5}{13}$. This gives us an adjacent side of 5 and a hypotenuse of 13. Using the Pythagorean Theorem, we get an opposite side of 12. So $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \frac{12}{5}$

38. -

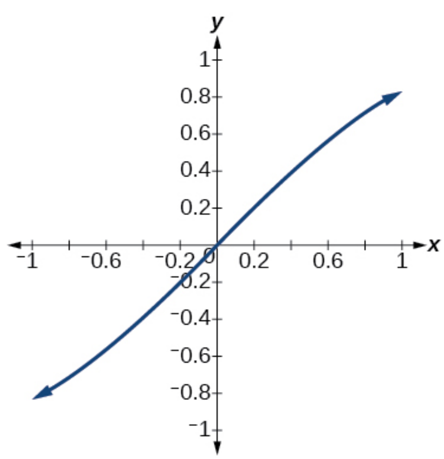
39. Graph $f(x) = \cos x$ and $f(x) = \sec x$ on the interval $[0, 2\pi)$ and explain any observations.

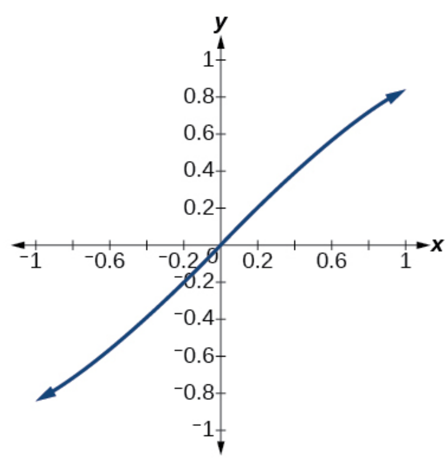


The graphs are not symmetrical with respect to the line $y = x$. They are symmetrical with respect to the y-axis. This means that they are not inverse functions of one another, and they are both even functions.

40. -

41. Graph the function $f(x) = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ on the interval $[-1,1]$ and compare the graph to the graph of $f(x) = \sin x$ on the same interval. Describe any observations.





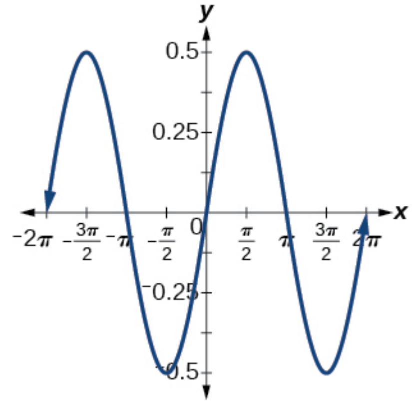
The graphs appear to be identical.

Chapter 8 Practice Test

For the following exercises, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

1. $f(x) = 0.5\sin x$

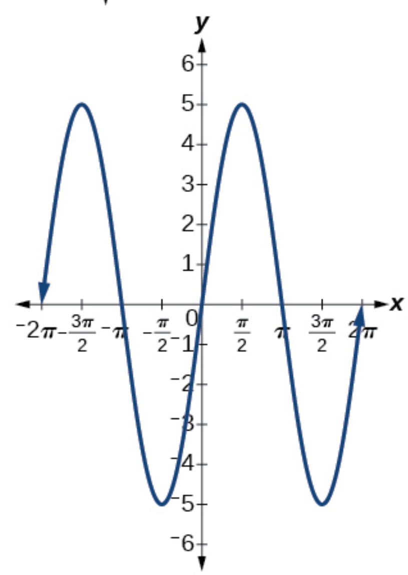
   $A = 0.5,\ B = 1,\ C = 0$ and $D = 0$. The amplitude is $|0.5| = 0.5$, so the function is compressed. The period is $\frac{2\pi}{1} = 2\pi$. There is no horizontal shift and the midline is $y = 0$.



2. -

3. $f(x) = 5\sin x$

   $A = 5,\ B = 1,\ C = 0$ and $D = 0$. The amplitude is $|5| = 5$, so the function is stretched by a factor of 5. The period is $\frac{2\pi}{1} = 2\pi$. There is no horizontal shift and the midline is $y = 0$.

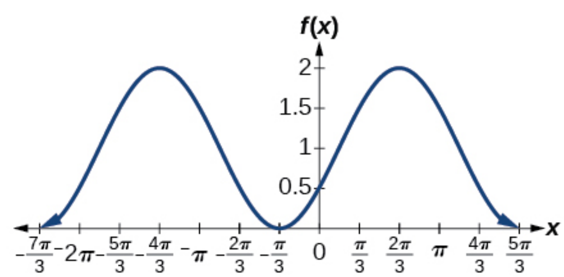


4. -

5. $f(x) = -\cos\left(x + \frac{\pi}{3}\right) + 1$

   $A = -1,\ B = 1,\ C = -\frac{\pi}{3}$ and $D = 1$. The amplitude is $|-1| = 1$, so the function is reflected across the midline. The period is $\frac{2\pi}{1} = 2\pi$. There is a horizontal shift of $\frac{\pi}{3}$ units to the left and the midline is $y = 0$.

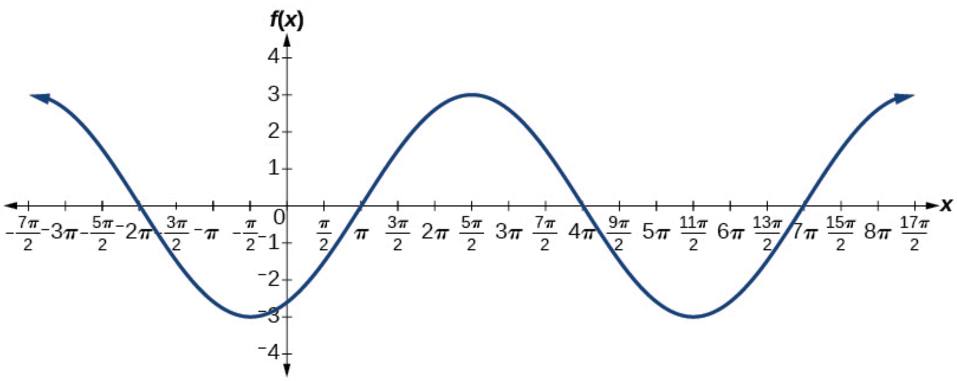
   amplitude: 1; period: 2π; midline: $y = 1$



6. -

7. $f(x) = 3\cos\left(\frac{1}{3}x - \frac{5\pi}{6}\right)$

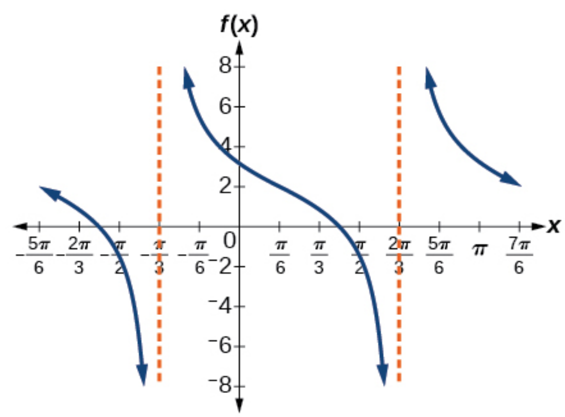
   $A = 3,\ B = \frac{1}{3},\ C = \frac{5\pi}{6}$ and $D = 0$. The amplitude is $|3| = 3$, so the function is stretched by a factor of 3. The period is $\frac{2\pi}{\frac{1}{3}} = 6\pi$. There is a horizontal shift of $\frac{\frac{5\pi}{6}}{\frac{1}{3}} = \frac{5\pi}{2}$ units to the right. The midline is $y = 0$.



8. -

9. $f(x) = -2\tan\left(x - \frac{7\pi}{6}\right) + 2$

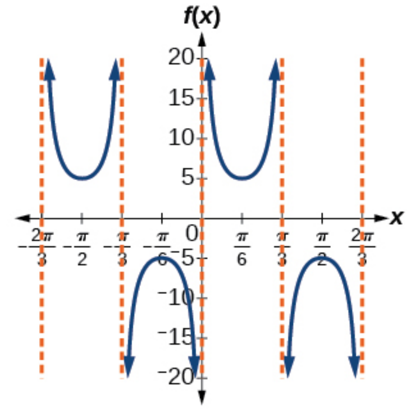
   $A = -2,\ B = 1,\ C = \frac{7\pi}{6}$ and $D = 2$. The stretching factor is $|-2| = 2$. The period is $\frac{\pi}{1} = \pi$. There is a horizontal shift of $\frac{7\pi}{6}$ units to the right. The vertical asymptotes occur at $x = \frac{\pi}{2} + \pi k + \frac{7\pi}{6} = \frac{5\pi}{3} + \pi k = \frac{2\pi}{3} + \pi k$, where k is some integer.



10. -

11. $f(x) = 5\csc(3x)$

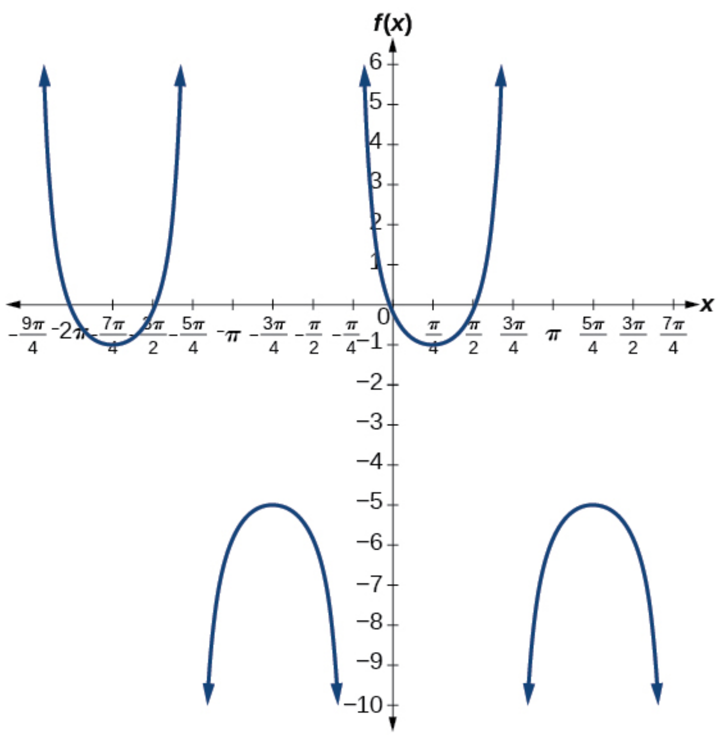
$A = 5,\ B = 3,\ C = 0$ and $D = 0$. The stretching factor is 5. The period is $\frac{2\pi}{3}$. There is no horizontal shift. The vertical asymptotes occur at $x = \frac{\pi}{3}k$, where k is an integer. The midline is $y = 0$.



12. -

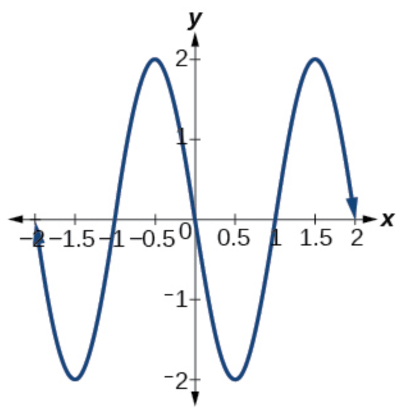
13. $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 3$

$A = 2,\ B = 1,\ C = -\frac{\pi}{4}$ and $D = -3$. The stretching factor is 2. The period is $\frac{2\pi}{1} = 2\pi$. There is a horizontal shift of $\frac{\pi}{4}$ units to the left. The vertical asymptotes occur at $x = \frac{\pi}{1}k = \pi k$, where k is an integer. The midline is $y = -3$.



For the following exercises, determine the amplitude, period, and midline of the graph, and then find a formula for the function.

14. -
15.



Give in terms of a sine function.
The midline is $y = 0$, so $D = 0$. The amplitude is 2, so $A = 2$ or –2. The function is at the midline when $x = 1$ and is increasing there, so we can write this with $A = 2$ and a horizontal shift of 1 unit to the right or with $A = -2$ and no horizontal shift. The period is 2 units. $\frac{2\pi}{B} = 2$ so $B = \pi$. With a horizontal shift of 1, $C = \pi$. This gives us the equation $f(x) = 2\sin(\pi(x-1))$ or $f(x) = -2\sin(\pi x)$

16. -

For the following exercises, find the amplitude, period, phase shift, and midline.

17. $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$

$A = 1,\ B = \frac{\pi}{6},\ C = -\pi$ and $D = -3$. The amplitude is 1. The period is $\frac{2\pi}{\frac{\pi}{6}} = 12$. The phase shift is $\frac{-\pi}{\frac{\pi}{6}} = -6$. and the midline is $y = -3$.

18. -
19. The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

The temperature is at its midline when $t = 0$, so we will write this as a sine function. The temperature is increasing overnight a total of 12 degrees, so $A = -12$. The period is

24 hours, so $\frac{2\pi}{B} = 24$ which gives us $B = \frac{\pi}{12}$. The midline is 68 degrees, so $D = 68$.

There is no phase shift, so $C = 0$. This gives us the function $D(t) = 68 - 12\sin\left(\frac{\pi}{12}x\right)$.

20. -

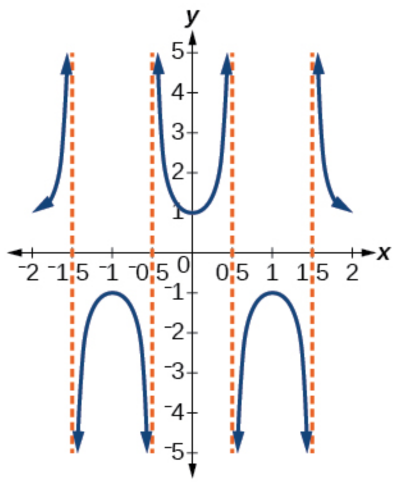
For the following exercises, find the period and horizontal shift of each function.

21. $g(x) = 3\tan(6x + 42)$

$A = 3,\ B = 6,\ C = -42$ and $D = 0$. The period is $\frac{\pi}{6}$. There is a horizontal shift of $\frac{-42}{6} = -7$; 7 units to the left.

22. -

23. Write the equation for the graph in the figure in terms of the secant function and give the period and phase shift.



The local minimum values are 1 and the local maximum values are –1, so the midline is $y = 0$. This gives us $D = 0$. The stretching factor is 1. The function is at a local minimum at $x = 0$, so $A = 1$ and the phase shift is 0. The period is 2, so $\frac{2\pi}{B} = 2$, and $B = \pi$. This gives us the equation $f(x) = \sec(\pi x)$.

24. -

25. If $\sec x = 4$, find $\sec(-x)$.

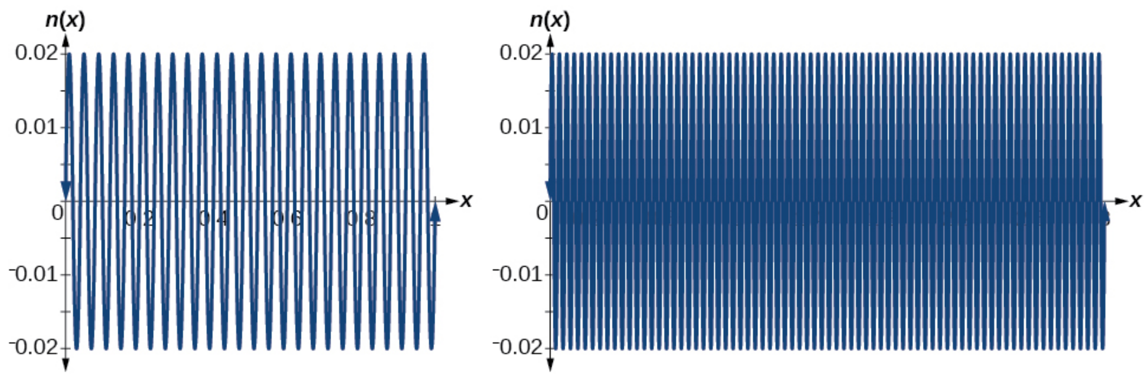
From the odd/even identities, we have $\sec(-x) = \sec x = 4$.

For the following exercises, graph the functions on the specified window and answer the questions.

26. -

The period is 2π or approximately 6.

27. -



The views are different because the period of the wave is 1/25. Over a bigger domain, there will be more cycles of the graph.

28. -.

For the following exercises, let $f\left(x\right)=\frac{3}{5}\cos\left(6x\right)$.

29. What is the largest possible value for $f\left(x\right)$?

The largest value for $\cos(6x)$ is 1, so the largest value for $f\left(x\right)=\frac{3}{5}\cos\left(6x\right)$ is $\frac{3}{5}$

30. -

31. Where is the function increasing on the interval $\left[0,2\pi\right]$?

The function has a period of $\frac{2\pi}{6}=\frac{\pi}{3}$ so it is increasing on the approximate intervals

$$\left(\frac{\pi}{6},\frac{\pi}{3}\right),\left(\frac{\pi}{2},\frac{2\pi}{3}\right),\left(\frac{5\pi}{6},\pi\right),\left(\frac{7\pi}{6},\frac{4\pi}{3}\right),\left(\frac{3\pi}{2},\frac{5\pi}{3}\right),\left(\frac{11\pi}{6},2\pi\right)$$

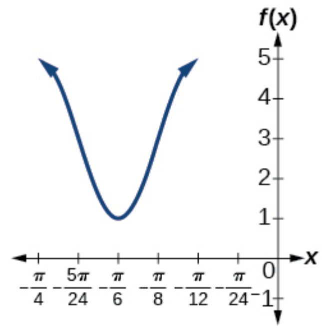
For the following exercises, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

32. -

33. Cosine curve with amplitude 2, period $\frac{\pi}{6}$, and shift $(h,k)=\left(-\frac{\pi}{4},3\right)$

The function has an amplitude of 2, so $A = 2$ or –2. The period is $\frac{\pi}{6}$, so $\frac{2\pi}{B}=\frac{\pi}{6}$, and $B=12$. There is a horizontal phase shift of $\frac{\pi}{4}$ units to the left. There is a vertical shift of 3 units up, so $D = 3$. Using $A = 2$, this gives us the equation

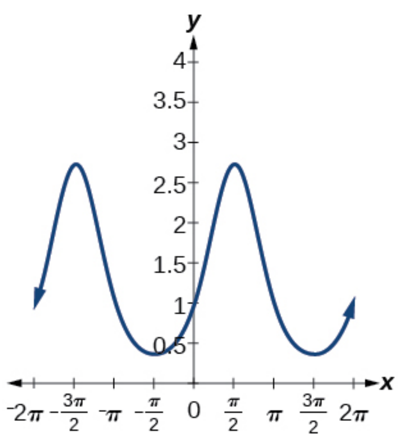
$$f(x)=2\cos\left(12\left(x+\frac{\pi}{4}\right)\right)+3$$



For the following exercises, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

34. -

35. $f(x)=e^{\sin t}$



This graph is periodic with a period of 2π.

For the following exercises, find the exact value.

36. -

37. $\tan^{-1}\left(\sqrt{3}\right)$

We need to find the angle, x, in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ such that $\tan x=\sqrt{3}$: $x=\frac{\pi}{3}$

38. -

39. $\cos^{-1}\left(\sin\left(\pi\right)\right)$

From the unit circle we have $\sin(\pi)=0,$ so we need to find the angle x in the interval $\left[0,\ \pi\right]$ such that $\cos x = 0$: $x=\dfrac{\pi}{2}$

40. -

41. $\cos\left(\sin^{-1}\left(1-2x\right)\right)$

We have an opposite side of $1-2x$ and a hypotenuse of 1. Using the Pythagorean Theorem, we get an adjacent side of $\sqrt{1-(1-2x)^2}$. So $\cos\left(\sin^{-1}\left(1-2x\right)\right)=\sqrt{1-(1-2x)^2}$

42. -

43. $\cos\left(\tan^{-1}\left(x^2\right)\right)$

We have an opposite side of x^2 and an adjacent side of 1. Using the Pythagorean Theorem, we get a hypotenuse of $\sqrt{1+x^4}$. This gives us $\cos(\tan^{-1}(x^2))=\dfrac{1}{\sqrt{1+x^4}}$

For the following exercises, suppose $\sin t = \dfrac{x}{x+1}$.

44. -

45. $\csc t$

From our result in number 44, we have $\csc t = \dfrac{x+1}{x}$

46. -

For the following exercises, determine whether the equation is true or false.

47. $\arcsin\left(\sin\left(\dfrac{5\pi}{6}\right)\right)=\dfrac{5\pi}{6}$

We have $\sin\left(\dfrac{5\pi}{6}\right)=\dfrac{1}{2}$, but for $\arcsin\left(\dfrac{1}{2}\right)$ we need an angle in the interval $\left[-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right]$, so this equation is false.

48. -

49. The grade of a road is 7%. This means that for every horizontal distance of 100 ft on the road, the vertical rise is 7 ft. Find the angle the road makes with the horizontal in radians.

We have an opposite side of 7 ft. and an adjacent side of 100 ft., so $\tan\theta=\dfrac{7}{100}$. We can use a calculator to find $\tan^{-1}(0.07)\approx 0.07$ radian.