

Appendix A Examples for Labs 1, 2, 3

1. FACTORING POLYNOMIALS

There are many standard methods of factoring that you have learned in previous courses. You will build on these factoring methods in your preCalculus course to enable you to factor expressions that will not factor by traditional methods. First you need to make sure you are skilled at the standard methods of factoring.

A. Greatest Common Factors

Example 1. Factor $8x^3y^4 - 6x^6y^2$

Since each term contains a factor $2x^3y^2$, factor out this GCF

$$\text{So } 8x^3y^4 - 6x^6y^2 = 2x^3y^2(4y^2 - 3x^3)$$

Example 2. Factor $3x^2(3x+4)^2 + 6x^3(3x+4)$

The common factor may contain a binomial term.

Each term contains a factor $3x^2(3x+4)$

$$\begin{aligned} \text{So } 3x^2(3x+4)^2 + 6x^3(3x+4) &= 3x^2(3x+4)[(3x+4) + 2x] \\ &= 3x^2(3x+4)(3x+4+2x) \\ &= 3x^2(3x+4)(5x+4) \end{aligned}$$

B. Difference of Squares $A^2 - B^2 = (A+B)(A-B)$

Example 1. Factor $49t^2 - 81v^2$

This one is easily recognizable as a difference of two perfect squares.

$$\text{So } 49t^2 - 81v^2 = (7t+9v)(7t-9v)$$

Example 2. Factor $625a^4 - b^8$

Make sure you completely factor the expression, remembering that the sum of squares does not factor.

$$\begin{aligned} 625a^4 - b^8 &= (25a^2 + b^4)(25a^2 - b^4) \\ &= (25a^2 + b^4)(5a + b^2)(5a - b^2) \end{aligned}$$

C. Sum or Difference of Cubes MEMORIZE!

$$\text{Formulas } A^3 + B^3 = (A+B)(A^2 - AB + B^2) \qquad A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

Example 1. Factor $125x^3 + 8$

$$\text{Use the formula for sum of cubes. } 125x^3 + 8 = (5x+2)(25x^2 - 10x + 4)$$

Example 2. Factor $64m^6 - 27y^6$

Use the formula for difference of cubes.

$$64m^6 - 27y^6 = (4m^2 - 3y)(16m^4 + 12m^2y + 9y^2)$$

D. Trinomials There are many methods of factoring general trinomials; usually a guess and check method is efficient. Think of the factoring as working backwards from a multiplication

Factor $6x^2 - x - 12$

By Guess and Check: $6x^2 - x - 12 = (3x + 4)(2x - 3)$

Check by multiplication: $(3x + 4)(2x - 3) = 6x^2 - 9x + 8x - 12$
 $= 6x^2 - x - 12$

5. Grouping strategies If an expression has four or more terms, we try grouping strategies to try to factor the expression.

Example 1: Factor $m^2 + 6m + 9 - p^2$

Think of this example as a difference of two perfect squares:

Grouping the terms first: $m^2 + 6m + 9 - p^2 = (m^2 + 6m + 9) - p^2$
 $= (m + 3)^2 - p^2$
 $= (m + 3 + p)(m + 3 - p)$

Example 2: Factor $x^3 - 3x^2 - 4x + 12$

The polynomial may contain a binomial common factor:

Grouping the terms in pairs: $x^3 - 3x^2 - 4x + 12 = (x^3 - 3x^2) - (4x - 12)$
 $= x^2(x - 3) - 4(x - 3)$
 $= (x - 3)(x^2 - 4)$
 $= (x - 3)(x + 2)(x - 2)$

2. RATIONAL EXPRESSIONS

Remember the properties of fractions, listed below to help you recall them. It is important to remember that division by zero is undefined over the real numbers, so no fraction can have a denominator of zero.

If a, b, c and d are real numbers and no denominator is zero, then:

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

$$\frac{ax}{bx} = \frac{a}{b}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

Skills in working with rational expressions are important to success in your preCalculus course. You will need to be able to add, subtract, multiply and divide rational

expressions. This includes simplifying complex fractions. The examples below will illustrate the procedures for simplifying rational expressions.

A. Simplification, multiplication, division

Always FACTOR first! NOTE: Only common factors cancel out, NOT terms! This is because canceling is really multiplication by 1.

$$\begin{aligned} \text{Example 1. } \frac{5x^2 - 20x}{4 - x} &= \frac{5x(x - 4)}{-1(x - 4)} && \text{(factor out -1 to see the common factor)} \\ &= -5x \end{aligned}$$

$$\begin{aligned} \text{Example 2 } \frac{2x^2 - 13x + 20}{x^2 - 16} \cdot \frac{2x^2 + 9x + 4}{6x^2 - 7x - 5} &= \frac{(2x - 5)(x - 4)}{(x + 4)(x - 4)} \cdot \frac{(2x + 1)(x + 4)}{(3x - 5)(2x + 1)} \\ &= \frac{2x - 5}{3x - 5} && \text{(don't cancel any more!)} \end{aligned}$$

$$\begin{aligned} \text{Example 3 } \frac{6x^2 - 3xy}{10mp^4} \div \frac{16x^2y^2 - 8xy^3}{15m^2p^2} &= \frac{3x(2x - y)}{10mp^4} \cdot \frac{15m^2p^2}{8xy^2(2x - y)} \\ &\quad \text{(multiply by reciprocal)} \\ &= \frac{9m}{16p^2y^2} \end{aligned}$$

B. Addition and subtraction

To add or subtract fractions, you must first find a common denominator, ideally the least common denominator, and then use the property $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$.

$$\begin{aligned} \text{Example 1: } \frac{2}{x-1} - \frac{1}{x+1} + \frac{2}{x^2-1} &= \frac{2}{x-1} - \frac{1}{x+1} + \frac{2}{(x+1)(x-1)} \\ \text{Write fraction with lcd} &= \frac{2(x+1)}{(x-1)(x+1)} - \frac{1(x-1)}{(x+1)(x-1)} + \frac{2}{(x+1)(x-1)} \\ \text{Add and subtract} &= \frac{2(x+1) - 1(x-1) + 2}{(x+1)(x-1)} \\ \text{Simplify} &= \frac{2x+2-x+1+2}{(x+1)(x-1)} \\ \text{Collect terms} &= \frac{x+5}{(x+1)(x-1)} \end{aligned}$$

Reduce to lowest terms if necessary. This example is already in lowest terms.

C. Complex Fractions

One method of simplifying a complex fraction is the following: simplify the numerator and the denominator to single fractions, and then divide. A complex fraction is really a division question.

$$\begin{aligned}
 \text{Example: } \frac{\frac{1}{y} - \frac{4y}{x^2}}{\frac{1}{y^2} + \frac{2}{xy}} &= \frac{\frac{x^2 - 4y^2}{x^2 y}}{\frac{x + 2y}{xy^2}} && \text{Write as a quotient of fractions} \\
 &= \frac{x^2 - 4y^2}{x^2 y} \div \frac{x + 2y}{xy^2} && \text{Division problem} \\
 &= \frac{x^2 - 4y^2}{x^2 y} \cdot \frac{xy^2}{x + 2y} && \text{Multiply by reciprocal} \\
 &= \frac{(x + 2y)(x - 2y)}{x^2 y} \cdot \frac{xy^2}{(x + 2y)} && \text{Factor and cancel} \\
 &= \frac{y(x - 2y)}{x}
 \end{aligned}$$

3. POLYNOMIAL MULTIPLICATION AND SPECIAL PRODUCTS

Multiplying polynomials often involves the special products shown below. You have used these patterns when factoring

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

$$\begin{aligned}
 \text{Example: } (5x + 3y)(5x - 3y) - (4x + y)^2 &= (25x^2 - 9y^2) - (16x^2 + 8xy + y^2) \\
 &= 25x^2 - 9y^2 - 16x^2 - 8xy - y^2 \\
 &= 9x^2 + 8xy - 10y^2
 \end{aligned}$$

4. ALGEBRAIC LONG DIVISION

In earlier examples, factoring was used to do polynomial division. When the division cannot be completed by factoring, algebraic long division can be used. Dividing polynomials algebraically uses the same procedure (algorithm) we use in arithmetic. The examples below will illustrate the procedures for division.

Example 1: Remember long division with whole numbers.

$$\begin{array}{r} 36 \\ 12 \overline{)438} \\ \underline{78} \\ 72 \\ \underline{6} \end{array}$$

The quotient is 36 and the remainder is 6. We usually write the answer as a mixed numeral or a decimal. $36\frac{6}{12} = 36\frac{1}{2} = 36.5$

Example 2: Dividing polynomials by factoring.

$$\begin{aligned} \frac{3x^2 + 11x + 6}{3x + 2} &= \frac{(3x + 2)(x + 3)}{(3x + 2)} \\ &= x + 3 \end{aligned}$$

Example 3: Using long division:

$$\begin{array}{r} x+3 \\ 3x+2 \overline{)3x^2+11x+6} \\ \underline{3x^2+2x} \\ 9x+6 \\ \underline{9x+6} \end{array}$$

The quotient is $x+3$

Example 4: $\frac{8x^3 - 30}{2x - 3} =$

$$\begin{array}{r} 4x^2 + 6x + 9 \\ 2x - 3 \overline{)8x^3 + 0x^2 + 0x - 30} \\ \underline{8x^3 - 12x^2} \\ 12x^2 + 0x \\ \underline{12x^2 - 18x} \\ 18x - 30 \\ \underline{18x - 27} \\ -3 \end{array}$$

Quotient notice the zero terms

Remainder

Answer: $4x^2 + 6x + 9 - \frac{3}{2x - 3}$

5. SOLVING EQUATIONS

When solving equations algebraically, it is often helpful to classify the type of equation you are solving so you can apply a suitable technique to that particular type of equation. The examples below will help you review some common equation solving techniques.

Example 1: Solving a linear equation

$$2(x - 3) = -2(x - 3)$$

$$2x - 6 = -2x + 6$$

$$2x + 2x = 6 + 6$$

$$4x = 12$$

$$x = 3$$

Simplify each side

Use opposites to collect variables to one side

Simplify and solve

The solution can be verified by substitution in the original equation.

Since division by zero is undefined, the variable cannot have a value that would require division by zero. Solutions must be checked for equations with variables in the denominator.

Example 2: Solve

$$\frac{3x}{x^2 + x} - \frac{2x}{x^2 + 5x} = \frac{x + 2}{x^2 + 6x + 5}$$

$$\frac{3x}{x(x+1)} - \frac{2x}{x(x+5)} = \frac{x+2}{(x+5)(x+1)}$$

Factor to find LCD

Determine restrictions ($x \neq 0, x \neq -1, x \neq -5$) since these values would make the denominator zero and the fractions would be undefined.

$$\left(\frac{3x}{x(x+1)} - \frac{2x}{x(x+5)} \right) \cdot (x(x+5)(x+1)) = \left(\frac{x+2}{(x+5)(x+1)} \right) \cdot (x(x+5)(x+1))$$

Multiply each side by LCD to clear fractions

$$3x(x+5) - 2x(x+1) = x(x+2)$$

Solve as in example 1

$$3x^2 + 15x - 2x^2 - 2x = x^2 + 2x$$

$$x^2 + 13x = x^2 + 2x$$

$$11x = 0$$

$$x = 0$$

But $x \neq 0$ so there is no real solution for this equation.

Often polynomial equations can be solved by a factoring method. The key to this method is what is referred to as the Zero Product rule, ie. if the product of two or more real numbers is zero, then at least one of the factors is equal to zero.

Example 3: Solve: $6x^2 = 4 - 5x$

$$6x^2 + 5x - 4 = 0$$

Set the equation equal to 0

$$(3x + 4)(2x - 1) = 0$$

Factor

$$x = -\frac{4}{3} \text{ or } x = \frac{1}{2}$$

Use zero product rule to solve

Example 4: Solve: $x^3 - 5x^2 - 9x + 45 = 0$
 $x^2(x-5) - 9(x-5) = 0$ Factor by grouping
 $(x-5)(x^2 - 9) = 0$
 $(x-5)(x+3)(x-3) = 0$
 $x = 5$ or $x = -3$ or $x = 3$ Use zero product rule to solve

The quadratic formula can be used to solve quadratic equation that will not factor, or even ones that do factor. The formula is derived from the general quadratic equation

$ax^2 + bx + c = 0$ where $a \neq 0$. The solutions are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Note that if $b^2 - 4ac < 0$, the radical is undefined and the equation has no solution.

Example 5: Solve $3x^2 - 5x - 1 = 0$
 Substitute the values $a = 3, b = -5, c = -1$ in the formula

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{37}}{6}$$

The two solutions are $\frac{5 + \sqrt{37}}{6}$ and $\frac{5 - \sqrt{37}}{6}$.

When solving equations involving radicals you must be sure to check your solution.

Example 6: Solve: $1 - \sqrt{2-x} = 2x$
 $1 - 2x = \sqrt{2-x}$ Isolate the radical
 $(1 - 2x)^2 = (\sqrt{2-x})^2$ Square each side
 $1 - 4x + 4x^2 = 2 - x$
 $4x^2 - 3x - 1 = 0$ Solve by factoring
 $(4x+1)(x-1) = 0$
 $x = -\frac{1}{4}$ or $x = 1$ Check solutions

Checking the two solutions shows that $x = -\frac{1}{4}$ is a solution to the original equation but

$x = 1$ is not. Therefore $x = -\frac{1}{4}$ is the only solution to the equation.

6. USING INTERVAL NOTATION

Intervals, sets of real numbers occurring between specified points, occur in solutions of inequalities (as well as other algebraic statements such as descriptions of the domain and range of functions). Interval notation is an efficient way of describing intervals. The table below shows examples of intervals. Remember that open end-points are designated by parentheses () and closed end-points by square brackets []

Set notation	Interval notation	Interval described in words
1. $\{x \mid x \leq 4\}$	$(-\infty, 4]$	from negative infinity up to and including 4
2. $\{x \mid x < 4\}$	$(-\infty, 4)$	from negative infinity up to but not including 4
3. $\{x \mid x > 4\}$	$(4, \infty)$	from 4 to infinity not including 4
4. $\{x \mid x \geq 4\}$	$[4, \infty)$	from 4 to infinity including 4
5. $\{x \mid -2 \leq x < 4\}$	$[-2, 4)$	from -2 to 4 including -2 but not 4
6. $\{x \mid -2 \leq x \leq 4\}$	$[-2, 4]$	from -2 to 4 including both -2 and 4
7. $\{x \mid -2 < x < 4\}$	$(-2, 4)$	from -2 to 4 excluding both -2 and 4
8. $\{x \mid -2 < x \leq 4\}$	$(-2, 4]$	from -2 to 4 excluding -2 but including 4
9. $\{x \mid x < -2 \text{ or } x \geq 4\}$	$(-\infty, -2) \cup [4, \infty)$	from negative infinity up to but not including -2 as well as from 4 to infinity including 4
10. $\{x \mid x \leq -2 \text{ or } x \geq 4\}$	$(-\infty, -2] \cup [4, \infty)$	from negative infinity up to and including -2 as well as from 4 to infinity including 4.

Interval Notation is much easier than words!

7. EXPONENTS

Recall the laws of exponents: If m , n and p are natural numbers, then:

Multiplication Law: $x^m x^n = x^{m+n}$

Division Law: $x^m \div x^n = x^{m-n}$

Power Laws: $(x^m)^n = x^{mn}$ $(x^m y^n)^p = x^{mp} y^{np}$ $\left(\frac{x^m}{y^n}\right)^p = \frac{x^{mp}}{y^{np}}$

By definition:

- If n is a natural number, then $x^n = x \cdot x \cdot x \cdots$ (n factors)
- If $x \neq 0$, $x^0 = 1$
- If $x \neq 0$, $x^{-1} = \frac{1}{x}$
- If $x \neq 0$ and n is a non-zero integer, then $x^{-n} = \frac{1}{x^n}$

- If $x \geq 0$ and n is a natural number, then $x^{\frac{1}{n}} = \sqrt[n]{x}$
- If $x \leq 0$ and n is an odd natural number, then $x^{\frac{1}{n}} = \sqrt[n]{x}$
- If $x \leq 0$ and n is an even natural number, then $x^{\frac{1}{n}}$ is not a real number.

NOTE: $x^{\frac{m}{n}}$ means $\left(x^{\frac{1}{n}}\right)^m$ or $\left(\sqrt[n]{x}\right)^m$. Therefore for $x^{m/n}$ to be a real number, $x^{1/n}$

must first be a real number. Some examples will help you sort out these definitions

$$\begin{aligned}\text{Example 1: } \frac{x^5 y^{-3}}{x^3 y} &= x^{5-3} \cdot x^{-3-1} \\ &= x^2 y^{-4} \\ &= \frac{x^2}{y^4}\end{aligned}$$

Sometimes you may want to simplify the expression inside the bracket first.

$$\begin{aligned}\text{Example 2: } \left(\frac{36x^{-2}y^3z^{-4}}{12xy^{-2}z^{-2}}\right)^{-2} &= \left(\frac{36}{12} \cdot x^{-2-1}y^{3-(-2)}z^{-4-(-2)}\right)^{-2} \\ &= \left(3 \cdot x^{-3}y^5z^{-2}\right)^{-2} \\ &= \left(\frac{3y^5}{x^3z^2}\right)^{-2} \\ &= \left(\frac{x^3z^2}{3y^5}\right)^2 \\ &= \frac{x^6z^4}{9y^{10}}\end{aligned}$$

Sometimes you may want to use the power rule first.

$$\begin{aligned}\text{Example 3: } \left(\frac{243a^{-5}}{32b^{-15}a^{10}}\right)^{\frac{2}{5}} &= \frac{243^{\frac{2}{5}}a^{-2}}{32^{\frac{2}{5}}b^{-6}a^4} \\ &= \frac{9}{4}a^{-2-4}b^{0-(-6)} \\ &= \frac{9}{4}a^{-6}b^6 \\ &= \frac{9b^6}{4a^6}\end{aligned}$$

Either procedure will give you the correct result. Try each example again, using the method of example 2 for example 3 and vice versa.

8. SIMPLIFYING RADICAL EXPRESSIONS

Basic Properties to remember:

If $a \geq 0, b \geq 0$, then

$$\sqrt[n]{a} = a^{\frac{1}{n}}, \quad a \geq 0$$

$$\left(\sqrt[n]{a}\right)^n = a$$

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, (b \neq 0)$$

The following examples illustrate how to express radical expression. in simplest form
For these examples we will assume all variables represent non-negative real numbers.

$$\begin{aligned} \text{Example 1} \quad \sqrt{175p^7q^8r^5} &= \sqrt{25 \cdot 7 \cdot p^6 \cdot p \cdot q^8 \cdot r^4 \cdot r} \\ &= 5p^3q^4r^2\sqrt{7pr} \end{aligned}$$

$$\begin{aligned} \text{Example 2:} \quad \sqrt[3]{81x^5y^6} &= \sqrt[3]{81x^5y^6} \\ &= \sqrt[3]{27 \cdot 3 \cdot x^3 \cdot x^2 \cdot y^6} \\ &= 3xy^2 \cdot \sqrt[3]{3x^2} \end{aligned}$$

$$\begin{aligned} \text{Example 3:} \quad \sqrt{98x^3y} + 3x\sqrt{32xy} - x\sqrt{50xy} &= \\ &= \sqrt{49 \cdot 2 \cdot x^2 \cdot x \cdot y} + 3x\sqrt{16 \cdot 2 \cdot x \cdot y} - x\sqrt{25 \cdot 2 \cdot x \cdot y} \\ &= 7x\sqrt{2xy} + 3x \cdot 4\sqrt{2xy} - x \cdot 5\sqrt{2xy} \\ &= 7x\sqrt{2xy} + 12x\sqrt{2xy} - 5x\sqrt{2xy} \\ &= 14x\sqrt{2xy} \end{aligned}$$

9. RATIONALIZING DENOMINATORS

Radicals in simplest form should not have a radical in the denominator. A procedure to eliminate the radical in a denominator is called rationalizing the denominator, ie. making it rational! We need to multiply the numerator and denominator by a radical that will make the denominator rational and simplify.

Example 1: For $x > 0$, simplify $\frac{3}{2\sqrt{x}}$

$$\begin{aligned} \frac{3}{2\sqrt{x}} &= \frac{3}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} \\ &= \frac{3\sqrt{x}}{2\sqrt{x^2}} \\ &= \frac{3\sqrt{x}}{2x} \end{aligned}$$

Example 2: Simplify: $\frac{3\sqrt{2} - 5\sqrt{3}}{6\sqrt{2} - 2\sqrt{3}}$

Multiply both the numerator and denominator by the conjugate of the denominator and simplify.

$$\begin{aligned}\frac{3\sqrt{2} - 5\sqrt{3}}{6\sqrt{2} - 2\sqrt{3}} &= \frac{(3\sqrt{2} - 5\sqrt{3})}{(6\sqrt{2} - 2\sqrt{3})} \cdot \frac{(6\sqrt{2} + 2\sqrt{3})}{(6\sqrt{2} + 2\sqrt{3})} \\ &= \frac{36 - 24\sqrt{6} - 30}{72 - 12} \\ &= \frac{6 - 24\sqrt{6}}{60} \\ &= \frac{1 - 4\sqrt{6}}{10}\end{aligned}$$

The next example uses the same procedure.

Example 3: Simplify: $\frac{\sqrt{x+h} - \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$ Assume all variables are positive

$$\begin{aligned}\frac{\sqrt{x+h} - \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} &= \frac{(\sqrt{x+h} - \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \cdot \frac{(\sqrt{x+h} - \sqrt{x})}{(\sqrt{x+h} - \sqrt{x})} \\ &= \frac{x+h - 2\sqrt{x(x+h)} + x}{x+h-x} \\ &= \frac{2x+h - 2\sqrt{x(x+h)}}{h}\end{aligned}$$