CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

Today’s Objectives:
Students will be able to:
1. Determine velocity and acceleration components using cylindrical coordinates.

In-Class Activities:
• Check Homework
• Reading Quiz
• Applications
• Velocity Components
• Acceleration Components
• Concept Quiz
• Group Problem Solving
• Attention Quiz
1. In a polar coordinate system, the velocity vector can be written as \( \mathbf{v} = v_t \mathbf{u}_r + v_\theta \mathbf{u}_\theta = \dot{r} \mathbf{u}_r + r\dot{\theta} \mathbf{u}_\theta \). The term \( \dot{\theta} \) is called
   A) transverse velocity.  B) radial velocity.
   C) angular velocity.  D) angular acceleration.

2. The speed of a particle in a cylindrical coordinate system is
   A) \( \dot{r} \)
   B) \( r\dot{\theta} \)
   C) \( \sqrt{(r\dot{\theta})^2 + (\dot{r})^2} \)
   D) \( \sqrt{(r\dot{\theta})^2 + (\dot{r})^2 + (\dot{z})^2} \)
The cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the boy slides down the slide at a constant speed of 2 m/s. How fast is his elevation from the ground changing (i.e., what is \( \dot{z} \) )?
A polar coordinate system is a 2-D representation of the cylindrical coordinate system.

When the particle moves in a plane (2-D), and the radial distance, $r$, is not constant, the polar coordinate system can be used to express the path of motion of the particle.
We can express the location of P in polar coordinates as $r = r u_r$. Note that the radial direction, $r$, extends outward from the fixed origin, $O$, and the transverse coordinate, $\theta$, is measured counterclockwise (CCW) from the horizontal.
VELOCITY (POLAR COORDINATES)

The instantaneous velocity is defined as:
\[ v = \frac{dr}{dt} = \frac{d(ru_r)}{dt} \]
\[ v = ru_r + r \frac{du_r}{dt} \]

Using the chain rule:
\[ \frac{du_r}{dt} = \left( \frac{du_r}{d\theta} \right) \frac{d\theta}{dt} \]
We can prove that \( \frac{du_r}{d\theta} = u_\theta \) so \( \frac{du_r}{dt} = \dot{\theta}u_\theta \)
Therefore:
\[ v = ru_r + r\dot{\theta}u_\theta \]

Thus, the velocity vector has two components: \( \mathbf{i} \), called the radial component, and \( r\dot{\theta} \), called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or
\[ v = \sqrt{(r \dot{\theta})^2 + (r \dot{r})^2} \]
The instantaneous acceleration is defined as:

\[ \mathbf{a} = \frac{dv}{dt} = (d/dt)(r \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta) \]

After manipulation, the acceleration can be expressed as

\[ \mathbf{a} = (\dot{r} - r \dot{\theta}^2) \mathbf{u}_r + (r \ddot{\theta} + 2r \dot{\theta}) \mathbf{u}_\theta \]

The term \((\dot{r} - r \dot{\theta}^2)\) is the radial acceleration or \(a_r\).

The term \((r \ddot{\theta} + 2r \dot{\theta})\) is the transverse acceleration or \(a_\theta\).

The magnitude of acceleration is

\[ a = \sqrt{(\dot{r} - r \dot{\theta}^2)^2 + (r \ddot{\theta} + 2r \dot{\theta})^2} \]
If the particle P moves along a space curve, its position can be written as

\[ \mathbf{r}_P = r \mathbf{u}_r + z \mathbf{u}_z \]

Taking time derivatives and using the chain rule:

Velocity: \[ \mathbf{v}_P = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta + \dot{z} \mathbf{u}_z \]

Acceleration: \[ \mathbf{a}_P = (\ddot{r} - r \dot{\theta}^2) \mathbf{u}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \mathbf{u}_\theta + \ddot{z} \mathbf{u}_z \]
EXAMPLE

Given: \( r = 5 \cos(2\theta) \) (m)
\[
\dot{\theta} = 3t^2 \text{ (rad/s)}
\]
\( \theta_0 = 0 \)

Find: Velocity and acceleration at \( \theta = 30^\circ \).

Plan: Apply chain rule to determine \( \dot{r} \) and \( \ddot{r} \) and evaluate at \( \theta = 30^\circ \).

Solution: \[
\theta = \int_{t_0}^{t} \dot{\theta} \, dt = \int_{0}^{t} 3t^2 \, dt = t^3
\]

At \( \theta = 30^\circ \), \[
\theta = \frac{\pi}{6} = t^3. \text{ Therefore: } t = 0.806 \text{ s.}
\]

\[
\dot{\theta} = 3t^2 = 3(0.806)^2 = 1.95 \text{ rad/s}
\]
EXAMPLE
(continued)

\[ \ddot{\theta} = 6t = 6(0.806) = 4.836 \text{ rad/s}^2 \]

\[ r = 5 \cos(2\theta) = 5 \cos(60) = 2.5 \text{ m} \]

\[ \dot{r} = -10 \sin(2\theta) \dot{\theta} = -10 \sin(60)(1.95) = -16.88 \text{ m/s} \]

\[ \ddot{r} = -20 \cos(2\theta) \dot{\theta}^2 - 10 \sin(2\theta) \ddot{\theta} \]

\[ = -20 \cos(60)(1.95)^2 - 10 \sin(60)(4.836) = -80 \text{ m/s}^2 \]

Substitute in the equation for velocity

\[ \mathbf{v} = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta \]

\[ \mathbf{v} = -16.88 \mathbf{u}_r + 2.5(1.95) \mathbf{u}_\theta \]

\[ \mathbf{v} = \sqrt{(16.88)^2 + (4.87)^2} = 17.57 \text{ m/s} \]
EXAMPLE
(continued)

Substitute in the equation for acceleration:

\[ a = (\ddot{r} - r\dot{\theta}^2)u_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})u_\theta \]

\[ a = [-80 - 2.5(1.95)^2]u_r + [2.5(4.836) + 2(-16.88)(1.95)]u_\theta \]

\[ a = -89.5u_r - 53.7u_\theta \text{ m/s}^2 \]

\[ a = \sqrt{(89.5)^2 + (53.7)^2} = 104.4 \text{ m/s}^2 \]
1. If \( \mathbf{r} \) is zero for a particle, the particle is
   A) not moving.       B) moving in a circular path.
   C) moving on a straight line.   D) moving with constant velocity.

2. If a particle moves in a circular path with constant velocity, its radial acceleration is
   A) zero.       B) \( \mathbf{i} \).
   C) \(-r\theta^2\).       D) \(2r\dot{\theta}\).
GROUP PROBLEM SOLVING

**Given:** The car’s speed is constant at 1.5 m/s.

**Find:** The car’s acceleration (as a vector).

**Hint:** The tangent to the ramp at any point is at an angle

\[ \phi = \tan^{-1}\left(\frac{12}{2\pi(10)}\right) = 10.81^\circ \]

Also, what is the relationship between \( \phi \) and \( \theta \)?

**Plan:** Use cylindrical coordinates. Since \( r \) is constant, all derivatives of \( r \) will be zero.

**Solution:** Since \( r \) is constant the velocity only has 2 components:

\[ v_\theta = r\dot{\theta} = v \cos\phi \quad \text{and} \quad v_z = \dot{z} = v \sin\phi \]
Therefore: \( \dot{\theta} = \left( \frac{v \cos \phi}{r} \right) = 0.147 \text{ rad/s} \)

\( \ddot{\theta} = 0 \)

\( v_z = \dot{z} = v \sin \phi = 0.281 \text{ m/s} \)

\( \ddot{z} = 0 \)

\( \ddot{r} = \dddot{r} = 0 \)

\[ a = (\dddot{r} - r\ddot{\theta}^2)u_r + (r\ddot{\theta} + 2r\dot{\theta})u_\theta + \ddot{z}u_z \]

\[ a = (-r\ddot{\theta}^2)u_r = -10(0.147)^2u_r = -0.217u_r \text{ m/s}^2 \]
1. The radial component of velocity of a particle moving in a circular path is always
   A) zero.
   B) constant.
   C) greater than its transverse component.
   D) less than its transverse component.

2. The radial component of acceleration of a particle moving in a circular path is always
   A) negative.
   B) directed toward the center of the path.
   C) perpendicular to the transverse component of acceleration.
   D) All of the above.