Chapter 5
Polynomial and Rational Functions
5.1 Quadratic Functions

Verbal

1. Explain the advantage of writing a quadratic function in standard form.
   When written in that form, the vertex can be easily identified.
2. -
3. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
   If $a = 0$ then the function becomes a linear function.
4. -
5. What two algebraic methods can be used to find the *x*- intercepts of a quadratic function?
   If possible, we can use factoring. Otherwise, we can use the quadratic formula.

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

6. -
7. $g(x) = x^2 + 2x - 3$

   The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{2}{2(1)} = -1.$

   The vertical coordinate of the vertex will be at
   $k = g(h) = g(-1) = (-1)^2 + 2(-1) - 3 = -4.$
   Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.
   $g(x) = ax^2 + bx + c$
   $g(x) = x^2 + 2x - 3$

   Using the vertex to determine the shifts, $g(x) = (x+1)^2 - 4$.
8. -
9. $f(x) = x^2 + 5x - 2$

   The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{5}{2(1)} = -\frac{5}{2}.$

   The vertical coordinate of the vertex will be at

   $k = f(h) = f\left(-\frac{5}{2}\right) = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) - 2 = -\frac{33}{4}.$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$f(x) = ax^2 + bx + c$

$f(x) = x^2 + 5x - 2$

Using the vertex to determine the shifts, $f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$.

10. -
11. $k(x) = 3x^2 - 6x - 9$

The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{-6}{2(3)} = 1$.

The vertical coordinate of the vertex will be at $k = k(1) = 3(1)^2 - 6(1) - 9 = -12$.

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$k(x) = ax^2 + bx + c$

$k(x) = 3x^2 - 6x - 9$

Using the vertex to determine the shifts, $h(x) = 3(x-1)^2 - 12$

12. -
13. $f(x) = 3x^2 - 5x - 1$

The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{-5}{2(3)} = \frac{5}{6}$.

The vertical coordinate of the vertex will be at $k = f\left(\frac{5}{6}\right) = 3\left(\frac{5}{6}\right)^2 - 5\left(\frac{5}{6}\right) - 1 = -\frac{37}{12}$.

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$f(x) = ax^2 + bx + c$

$f(x) = 3x^2 - 5x - 1$

Using the vertex to determine the shifts, $f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12}$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

14. -

15. $f(x) = 2x^2 - 10x + 4$

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-10}{2(2)} \\ &= \frac{5}{2} \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{5}{2}\right) &= 2\left(\frac{5}{2}\right)^2 - 10\left(\frac{5}{2}\right) + 4 \\ &= -\frac{17}{2} \end{aligned}$$

Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = \frac{5}{2}$.

16. -

17. $f(x) = 4x^2 + x - 1$

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{1}{2(4)} \\ &= -\frac{1}{8} \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f\left(-\frac{1}{8}\right) &= 4\left(-\frac{1}{8}\right)^2 + \left(-\frac{1}{8}\right) - 1 \\ &= -\frac{17}{16} \end{aligned}$$

Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = -\frac{1}{8}$.

18. -

19. $f(x)=\frac{1}{2}x^2+3x+1$

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{3}{2\left(\frac{1}{2}\right)} \\ &= -3 \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f(-3) &= \frac{1}{2}(-3)^2+3(-3)+1 \\ &= -\frac{7}{2} \end{aligned}$$

Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x=-3$.

20. -

For the following exercises, determine the domain and range of the quadratic function.

21. $f(x)=(x-3)^2+2$

The general form of the given quadratic function is $f(x)=x^2-6x+11$.

As with any quadratic function, the domain is all real numbers, that is $(-\infty,\infty)$.

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(1)} \\ &= 3 \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f(3) &= (3-3)^2+2 \\ &= 2 \end{aligned}$$

The range is $f(x)\geq 2$ or $[2,\infty)$.

22. -

23. $f(x)=x^2+6x+4$

As with any quadratic function, the domain is all real numbers, that is $(-\infty,\infty)$.

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{6}{2(1)} \\ &= -3 \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f(-3) &= (-3)^2 + 6\left(-3\right) + 4 \\ &= -5 \end{aligned}$$

The range is $f(x) \ge -5$ or $[-5, \infty)$.

24. -

25. $k\left(x\right) = 3x^2 - 6x - 9$

As with any quadratic function, the domain is all real numbers, that is $\left(-\infty, \infty\right)$.

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(3)} \\ &= 1 \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f(1) &= 3(1)^2 - 6\left(1\right) - 9 \\ &= -12 \end{aligned}$$

The range is $f(x) \ge -12$ or $[-12, \infty)$.

For the following exercises, use the vertex $\left(h,\ k\right)$ and a point on the graph $\left(x,\ y\right)$ to find the general form of the equation of the quadratic function.

26. -

27. $\left(h,\ k\right) = \left(-2, -1\right)$, $\left(x,\ y\right) = \left(-4,\ 3\right)$

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x+2)^2 - 1 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(-4,\ 3)$, we can solve for the stretch factor.

$3 = a(-4+2)^2 - 1$

$3 = 4a - 1$

$4 = 4a$

$a = 1$

In standard form, the algebraic model for this graph is $f(x) = (x+2)^2 - 1$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= (x+2)^2 - 1 \\ &= (x+2)(x+2) - 1 \\ &= x^2 + 4x + 4 - 1 \\ &= x^2 + 4x + 3 \end{aligned}$$

28. -

29. $(h,\ k) = (2,\ 3),\ (x,\ y) = (5,\ 12)$

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x-2)^2 + 3 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(5,\ 12)$, we can solve for the stretch factor.

$12 = a(5-2)^2 + 3$

$12 = 9a + 3$

$9 = 9a$

$a = 1$

In standard form, the algebraic model for this graph is $f(x) = (x-2)^2 + 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= (x-2)^2 + 3 \\ &= (x-2)(x-2) + 3 \\ &= x^2 - 4x + 4 + 3 \\ &= x^2 - 4x + 7 \end{aligned}$$

30. -

31. $(h,\ k) = (3,\ 2),\ (x,\ y) = (10,\ 1)$

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2+k \\ &= a(x-3)^2+2 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(10,\ 1)$, we can solve for the stretch factor.

$$\begin{aligned} 1 &= a(10-3)^2+2 \\ 1 &= 49a+2 \\ -1 &= 49a \\ a &= -\frac{1}{49} \end{aligned}$$

In standard form, the algebraic model for this graph is $f(x) = -\frac{1}{49}(x-3)^2+2.$

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= -\frac{1}{49}(x-3)^2+2 \\ &= -\frac{1}{49}(x-3)(x-3)+2 \\ &= -\frac{1}{49}\left(x^2-6x+9\right)+2 \\ &= -\frac{1}{49}x^2+\frac{6}{49}x+\frac{89}{49} \end{aligned}$$

32. -

33. $(h,\ k) = (1,\ 0),\ (x,\ y) = (0,\ 1)$

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2+k \\ &= a(x-1)^2+0 \\ &= a(x-1)^2 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(0,\ 1)$, we can solve for the stretch factor.

$$\begin{aligned} 1 &= a(0-1)^2 \\ 1 &= a\cdot 1 \\ a &= 1 \end{aligned}$$

In standard form, the algebraic model for this graph is $f(x) = (x-1)^2$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= (x-1)^2 \\ &= (x-1)(x-1) \\ &= x^2 - 2x + 1 \end{aligned}$$

Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

34. -

35. $f(x) = x^2 - 6x - 1$

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the *x*-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2 \cdot 1} \\ &= 3 \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f(3) &= 3^2 - 6(3) - 1 \\ &= -10 \end{aligned}$$

The vertex is at $(3, -10)$.

Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$.

We find the *y*- intercept by evaluating $f(0)$.

$$f(0) = (0)^2 - 6(0) - 1 = -1$$

So the *y*- intercept is at $(0, -1)$.

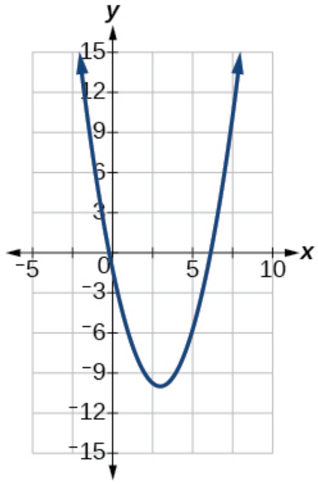
For the *x*-intercepts, we find all solutions of $f(x) = 0$.

$$0 = x^2 - 6x - 1$$

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2 - 6x - 1 = 0$, we have $a = 1$, $b = -6$, and $c = -1$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \\ &= \frac{6 \pm \sqrt{40}}{2} \\ &= \frac{6 \pm 2\sqrt{10}}{2} \\ &= 3 \pm \sqrt{10} \end{aligned}$$

So the x-intercepts are at $\left(3+\sqrt{10},\ 0\right)$ and $\left(3-\sqrt{10},\ 0\right)$.



36. -

37. $f(x) = x^2 - 7x + 3$

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-7}{2 \cdot 1} \\ &= \frac{7}{2} \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{7}{2}\right) &= \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) + 3 \\ &= -\frac{37}{4} \end{aligned}$$

The vertex is at $\left(\frac{7}{2}, -\frac{37}{4}\right)$.

Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = \frac{7}{2}$.

We find the *y*- intercept by evaluating $f(0)$.

$f(0) = (0)^2 - 7(0) + 3 = 3$

So the *y*- intercept is at $(0,\ 3)$.

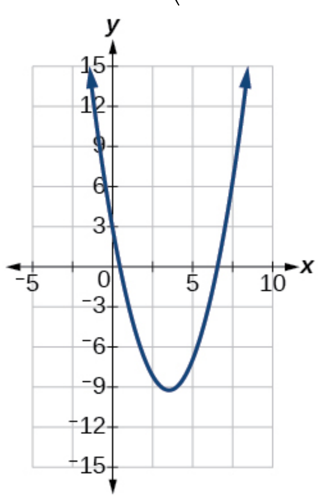
For the *x*-intercepts, we find all solutions of $f(x) = 0$.

$0 = x^2 - 7x + 3$

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2 - 7x + 3 = 0$, we have $a = 1$, $b = -7$, and $c = 3$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{7 \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \\ &= \frac{7 \pm \sqrt{37}}{2} \end{aligned}$$

So the *x*-intercepts are at $\left(\frac{7+\sqrt{37}}{2},\ 0\right)$ and $\left(\frac{7-\sqrt{37}}{2},\ 0\right)$.



38. -

39. $f(x) = 4x^2 - 12x - 3$

Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-12}{2 \cdot 4} \\ &= \frac{3}{2} \end{aligned}$$

The minimum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{3}{2}\right) &= 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) - 3 \\ &= -12 \end{aligned}$$

The vertex is at $\left(\frac{3}{2}, -12\right)$.

Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = \frac{3}{2}$.

We find the y- intercept by evaluating $f(0)$.

$f(0) = (0)^2 - 12(0) - 3 = -3$

So the y- intercept is at $(0, -3)$.

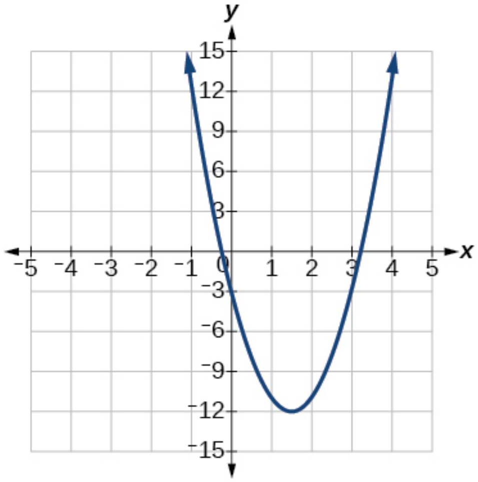
For the x-intercepts, we find all solutions of $f(x) = 0$.

$0 = 4x^2 - 12x - 3$

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $4x^2 - 12x - 3 = 0$, we have $a = 4$, $b = -12$, and $c = -3$. Substituting these values into the formula we have:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{12 \pm \sqrt{(-12)^2 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4} \\
&= \frac{12 \pm \sqrt{192}}{8} \\
&= \frac{12 \pm 8\sqrt{3}}{8} \\
&= \frac{3 \pm 2\sqrt{3}}{2}
\end{aligned}
$$

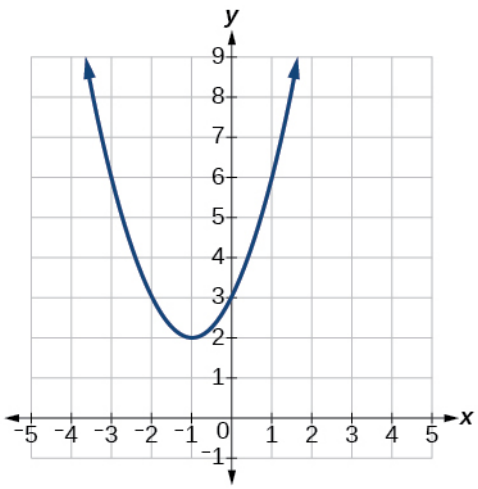
So the x-intercepts are at $\left(\frac{3+2\sqrt{3}}{2},\ 0\right)$ and $\left(\frac{3-2\sqrt{3}}{2},\ 0\right)$.



For the following exercises, write the equation for the graphed function.

40. -

41.



The vertex is the turning point of the graph. We can see that the vertex (h, k) is located at $(-1, 2)$.

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x+1)^2 + 2 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(1,\ 6)$, we can solve for the stretch factor.

$$6 = a(1+1)^2 + 2$$
$$6 = 4a + 2$$
$$4 = 4a$$
$$a = 1$$

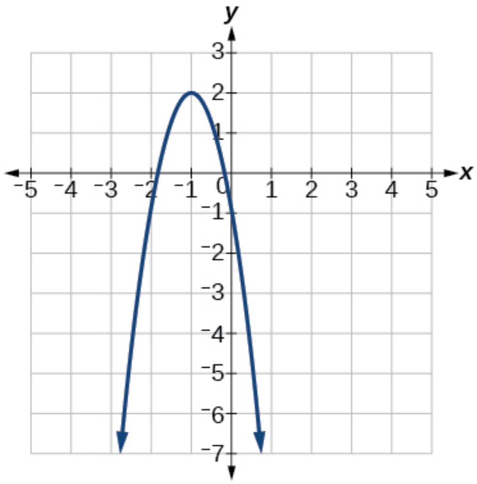
In standard form, the algebraic model for this graph is $f(x) = (x+1)^2 + 2.$

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= (x+1)^2 + 2 \\ &= (x+1)(x+1) + 2 \\ &= x^2 + 2x + 1 + 2 \\ &= x^2 + 2x + 3 \end{aligned}$$

42. -

43.



The vertex is the turning point of the graph. We can see that the vertex (h, k) is located at $(-1, 2)$.

Substitute the values of h and k in the function

$$f(x) = a(x-h)^2 + k$$
$$= a(x+1)^2 + 2$$

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+1)^2 + 2$$
$$-1 = a + 2$$
$$a = -3$$

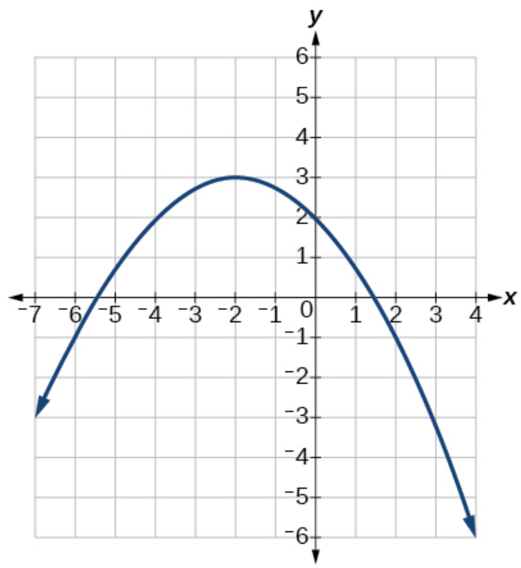
In standard form, the algebraic model for this graph is $f(x) = -3(x+1)^2 + 2.$

To write this in general polynomial form, we can expand the formula and simplify terms.

$$f(x) = -3(x+1)^2 + 2$$
$$= -3(x+1)(x+1) + 2$$
$$= -3\left(x^2 + 2x + 1\right) + 2$$
$$= -3x^2 - 6x - 1$$

44. -

45.



The vertex is the turning point of the graph. We can see that the vertex (h, k) is located at $(-2, 3)$.

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x+2)^2 + 3 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(-4, 2)$, we can solve for the stretch factor.

$$\begin{aligned} 2 &= a(-4+2)^2 + 3 \\ 2 &= 4a + 3 \\ -1 &= 4a \\ a &= -\frac{1}{4} \end{aligned}$$

In standard form, the algebraic model for this graph is $f(x) = -\frac{1}{4}(x+2)^2 + 3.$

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= -\frac{1}{4}(x+2)^2 + 3 \\ &= -\frac{1}{4}(x+2)(x+2) + 3 \\ &= -\frac{1}{4}\left(x^2 + 4x + 4\right) + 3 \\ &= -\frac{1}{4}x^2 - x + 2 \end{aligned}$$

Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

46. -

47.

x	–2	–1	0	1	2
y	1	0	1	4	9

The vertex is the turning point of the quadratic function. From the table, vertex (h, k) is located at $(-1, 0)$.

Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x+1)^2 + 0 \\ &= a(x+1)^2 \end{aligned}$$

Substituting the coordinates of a point on the curve, such as $(0,\ 1)$, we can solve for the stretch factor.

$$1 = a(0+1)^2$$
$$1 = a \cdot 1$$
$$a = 1$$

In standard form, the algebraic model for this graph is $f(x) = (x+1)^2$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} f(x) &= (x+1)^2 \\ &= (x+1)(x+1) \\ &= x^2 + 2x + 1 \end{aligned}$$

48. -

49.

x	–2	–1	0	1	2
y	–8	–3	0	1	0

The vertex is the turning point of the quadratic function. From the table, vertex (h, k) is located at $(1, 1)$.

Substitute the values of h and k in the function

$$f(x) = a(x-h)^2 + k$$
$$= a(x-1)^2 + 1$$

Substituting the coordinates of a point on the curve, such as $(0,\ 0)$, we can solve for the stretch factor.

$$0 = a(0-1)^2 + 1$$
$$a = -1$$

In standard form, the algebraic model for this graph is $f(x) = -1(x-1)^2 + 1.$

To write this in general polynomial form, we can expand the formula and simplify terms.

$$f(x) = -1(x-1)^2 + 1$$
$$= -(x-1)(x-1) + 1$$
$$= -\left(x^2 - 2x + 1\right) + 1$$
$$= -x^2 + 2x$$

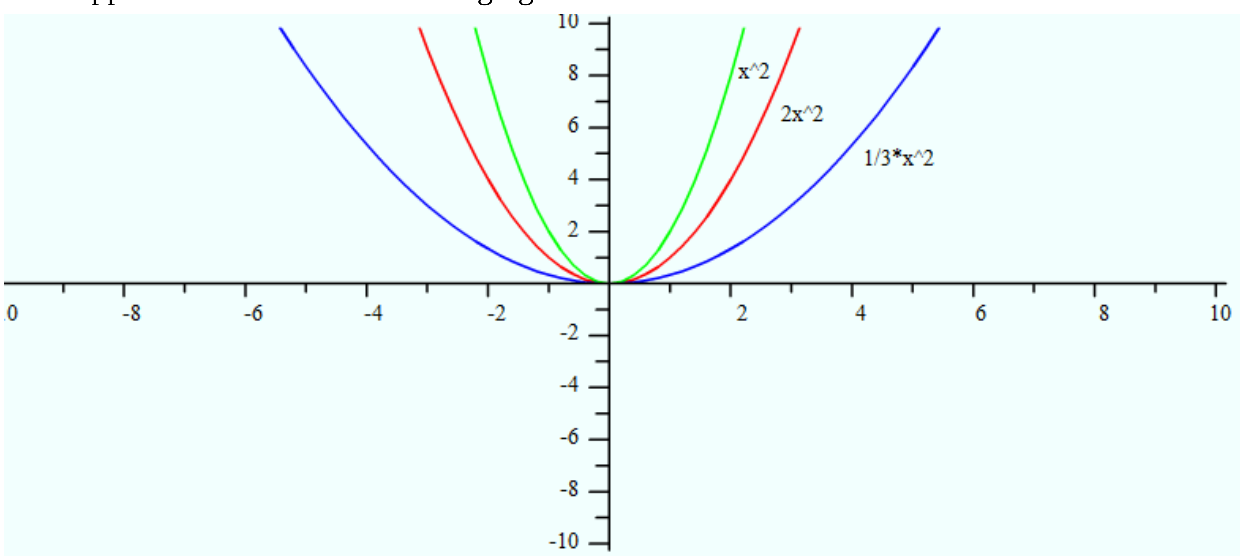
50. -

Technology

For the following exercises, use a calculator to find the answer.

51. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$.

What appears to be the effect of changing the coefficient?

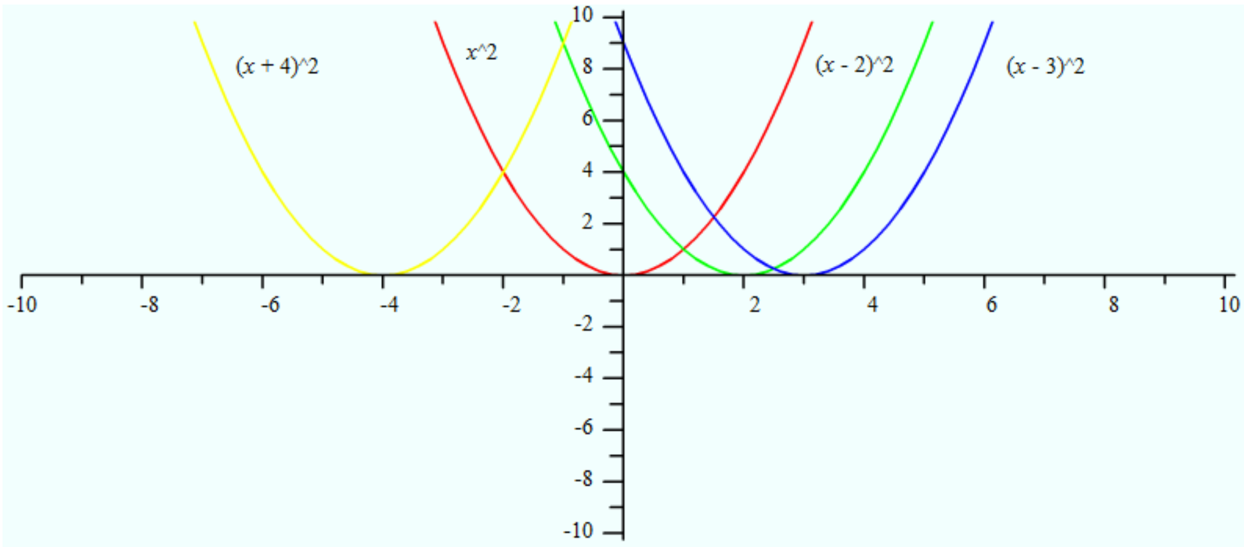


The value stretches or compresses the width of the graph. The greater the value, the narrower the graph.

52. -

53. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x-2)^2$, $f(x-3)^2$, and $f(x) = (x+4)^2$.

What appears to be the effect of adding or subtracting those numbers?



The graph is shifted to the right or left (a horizontal shift).

54. -

55. A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \le x \le 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Enter the function as Y1.

Get the $\boxed{\text{TRACE}}$ function and press the right arrow repeatedly until the new type of cursor gives a y value as close to 100. The closest point is (1000, 100).

The suspension bridge has 1000 feet distance from the center.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

56. -

57. Vertex $(-1, 2)$, opens down.

As with any quadratic function, the domain is all real numbers, that is $(-\infty, \infty)$.

If the parabola opens down, the vertex represents the maximum value of the quadratic function.

A quadratic function's maximum value is given by the *y*-value of the vertex.

Thus, the range is given by $f(x) \le 2$, or $(-\infty, 2]$.

58. -

59. Vertex $(-100, 100)$, opens up.

As with any quadratic function, the domain is all real numbers, that is $(-\infty, \infty)$.

If the parabola opens up, the vertex represents the minimum value of the quadratic function.

A quadratic function's minimum value is given by the *y*-value of the vertex.

Thus, the range is given by $f(x) \ge 100$, or $[100, \infty)$.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

60. -

61. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the *y*-axis.

The equation of the quadratic function that contains the given point and has the same shape as the given function is $f(x) = 2x^2 + c$.

Substitute $x = -1$ and $f(x) = 4$ in the function

$$f(x) = 2x^2 + c$$
$$4 = 2 \cdot (-1)^2 + c$$
$$c = 2$$

Thus, the required equation is $f(x) = 2x^2 + 2$.

62. -

63. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the *y*-axis.

The equation of the quadratic function that contains the given point and has the same shape as the given function is $f(x) = -x^2 + c$.

Substitute $x = 1$ and $f(x) = -3$ in the function

$$f(x) = -x^2 + c$$
$$-3 = -1^2 + c$$
$$c = -2$$

Thus, the required equation is $f(x) = -x^2 - 2$.

64. -

65. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

The equation of the quadratic function that contains the given point and has the same shape as the given function is $f(x) = 3(x-(-1))^2 + c = 3(x+1)^2 + c$

Substitute $x = 1$ and $f(x) = -6$ in the function

$$f(x) = 3(1+1)^2 + c$$
$$-6 = 3\cdot 4 + c$$
$$c = -18$$

Thus, the required equation is

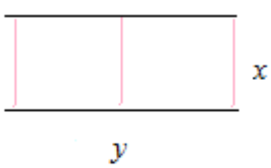
$$\begin{aligned} f(x) &= 3(x+1)^2 - 18 \\ &= 3(x^2 + 2x + 1) - 18 \\ &= 3x^2 + 6x + 3 - 18 \\ &= 3x^2 + 6x - 15 \end{aligned}$$

Real-World Applications

66. -

67. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

The diagram shows the rectangular corral producing the greatest enclosed area split into 3 pens of the same size.



So, we have 3 sections of fencing with length *x* and 2 sections of fencing with length *y*.

$$3x + 2y = 300$$
$$2y = 300 - 3x$$
$$y = 150 - \frac{3}{2}x$$

Total area = *xy*

Express area as a function of x if we substitute y with $150-\frac{3}{2}x$.

$$f(x)=x\left(150-\frac{3}{2}x\right)$$
$$=-\frac{3}{2}x^2+150x$$

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value.

We can begin by finding the x-value of the vertex.

$$h=-\frac{b}{2a}$$
$$=-\frac{150}{2\cdot\left(-\frac{3}{2}\right)}$$
$$=50$$

The maximum value of y is given by

$$y=150-\frac{3}{2}(50)$$
$$=150-75$$
$$=75$$

The rectangular corral producing the greatest enclosed area split into 2 pens of the same size given 300 feet of fencing is 75 feet by 50 feet.

68. -

69. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

Let x be one number and $6-x$ be another number.

Then their product will be $f(x)=x(6-x)=-x^2+6x$.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value.

We can begin by finding the x-value of the vertex.

$$h=-\frac{b}{2a}$$
$$=-\frac{6}{2\cdot(-1)}$$
$$=3$$

The maximum value is given by $f(h)$.

$$f(3) = -(3)^2 + 6(3)$$
$$= 9$$

Thus, the pair with the largest product is 3 and $6 - 3 = 3$ and their product is 9.

70. -

71. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

Substitute $p = 45 - 0.0125x$ into R.

$$R = x(45 - 0.0125x)$$
$$= -0.0125x^2 + 45x$$

The revenue reaches the maximum value at the vertex of the parabola.

$$h = -\frac{45}{2(-0.0125)}$$
$$= 1800$$

The revenue reaches the maximum value when 1800 thousand phones are produced.

72. -

73. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{24}{2(-4.9)}$$
$$= \frac{24}{9.8}$$
$$\approx 2.449$$

The ball reaches a maximum height after 2.449 seconds.

74. -

75. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

Let x be the additional tree planted per acre. Then, the fruit per acre should be

$$\begin{aligned} f(x) &= (75+x)(20-3x) \\ &= 1500 - 225x + 20x - 3x^2 \\ &= 1500 - 205x - 3x^2 \end{aligned}$$

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value.
We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{(-205)}{2(-3)} \\ &= -\frac{205}{6} \\ &\approx -34 \end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(-34\right) &= 1500 - 205\cdot 34 - 3\cdot 34^2 \\ &\approx 5002 \end{aligned}$$

So, the vertex is at $(-34, 5002)$.
She should plant $75 - 34 = 41$ trees per acre to maximize her yield.

Chapter 5
Polynomial and Rational Functions
5.2 Power Functions and Polynomial Functions

Verbal

1. Explain the difference between the coefficient of a power function and its degree.
   The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.
2. -
3. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.
   As x decreases without bound, so does $f(x)$. As x increases without bound, so does $f(x)$.
4. -
5. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.
   The polynomial function is of even degree and leading coefficient is negative.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

6. -
7. $f(x) = \left(x^2\right)^3$
   $f(x)$ is a power function because it contains a variable base raised to a fixed power.
8. -
9. $f(x) = \dfrac{x^2}{x^2 - 1}$
   $f(x)$ is neither a polynomial function nor a power function because it does not consist of the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power and it does not contain a variable base raised to a fixed power.
10. -
11. $f(x) = 3^{x+1}$
    $f(x)$ is neither a polynomial function nor a power function because it does not consist of the sum of a finite number of non-zero terms, each of which is a product of a number,

called the coefficient of the term, and a variable raised to a non-negative integer power and it does not contain a variable base raised to a fixed power.

For the following exercises, find the degree and leading coefficient for the given polynomial.

12. -

13. $7-2x^2$

The highest power of x is 2, so the degree is 2. The leading term is the term containing that degree, $-2x^2$. The leading coefficient is the coefficient of that term, -2.

14. -

15. $x\left(4-x^2\right)(2x+1)$

Obtain the general form by expanding the given expression.

$$\begin{aligned} x\left(4-x^2\right)(2x+1) &= x\left(8x+4-2x^3-x^2\right) \\ &= 8x^2+4x-2x^4-x^3 \end{aligned}$$

The highest power of x is 4, so the degree is 4. The leading term is the term containing that degree, $-2x^4$. The leading coefficient is the coefficient of that term, -2.

16. -

For the following exercises, determine the end behavior of the functions.

17. $f\left(x\right)=x^4$

The coefficient is 1 (positive) and the exponent of the power function is 4 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x\to\infty,\ f(x)\to\infty.$ As x approaches negative infinity, the output increases without bound. In symbolic form, as $x\to-\infty, f(x)\to\infty.$

18. -

19. $f\left(x\right)=-x^4$

The coefficient is -1 (negative) and the exponent of the power function is 4 (an even number). As x approaches infinity, the output (value of $f(x)$) decreases without bound. We write as $x\to\infty, f(x)\to-\infty.$ As x approaches negative infinity, the output decreases without bound. In symbolic form, as $x\to-\infty, f(x)\to-\infty.$

20. -

21. $f(x)=-2x^4-3x^2+x-1$

The leading term is $-2x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–2), so the end behavior is

as $x \to -\infty, f(x) \to -\infty$
as $x \to \infty, f(x) \to -\infty$.

22. -

23. $f(x) = x^2(2x^3 - x + 1)$

Obtain the general form by expanding the given expression for $f(x)$.

$$f(x) = x^2(2x^3 - x + 1)$$
$$= 2x^5 - x^3 + x^2$$

The general form is $f(x) = 2x^5 - x^3 + x^2$. The leading term is $2x^5$; therefore, the degree of the polynomial is 5. The degree is odd (5) and the leading coefficient is positive (2), so the end behavior is

as $x \to \infty, f(x) \to \infty$
as $x \to -\infty, f(x) \to -\infty$.

24. -

For the following exercises, find the intercepts of the functions.

25. $f(t) = 2(t-1)(t+2)(t-3)$

The y-intercept occurs when the input is zero so substitute 0 for t.

$$f(0) = 2(0-1)(0+2)(0-3)$$
$$= 2(-1)(2)(-3)$$
$$= 12$$

The y-intercept is (0, 12).

The t-intercepts occur when the output is zero.

$$0 = 2(t-1)(t+2)(t-3)$$
$$t - 1 = 0 \quad \text{or} \quad t + 2 = 0 \quad \text{or} \quad t - 3 = 0$$
$$t = 1 \quad \text{or} \quad t = -2 \quad \text{or} \quad t = 3$$

The t-intercepts are $(1,0)$, $(-2,0)$, and $(3,0)$.

26. -

27. $f(x) = x^4 - 16$

The y-intercept occurs when the input is zero so substitute 0 for x.

$$f(0) = 0^4 - 16$$
$$= -16$$

The y-intercept is $(0, -16)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 16 \\ &= (x^2 - 4)(x^2 + 4) \\ &= (x-2)(x+2)(x^2 + 4) \end{aligned}$$

$$0 = (x-2)(x+2)(x^2 + 4)$$

$x - 2 = 0$ or $x + 2 = 0$ or $x^2 + 4 = 0$

$x = 2$ or $x = -2$ or (no real solution)

The x-intercepts are $(2, 0)$ and $(-2, 0)$.

28. -

29. $f(x) = x\left(x^2 - 2x - 8\right)$

The y-intercept occurs when the input is zero so substitute 0 for x.

$$\begin{aligned} f(0) &= 0\left(0^2 - 2 \cdot 0 - 8\right) \\ &= 0 \end{aligned}$$

The y-intercept is (0, 0).

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x\left(x^2 - 2x - 8\right) \\ &= x(x-4)(x+2) \end{aligned}$$

$$0 = x(x-4)(x+2)$$

$x = 0$ or $x - 4 = 0$ or $x + 2 = 0$

$x = 4$ or $x = -2$

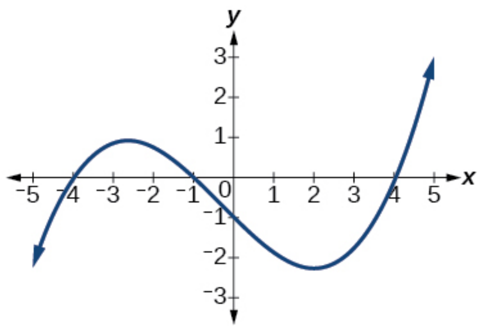
The x-intercepts are $(0, 0)$, $(4, 0)$, and $(-2, 0)$.

30. -

Graphical

For the following exercises, determine the least possible degree of the polynomial function shown.

31.

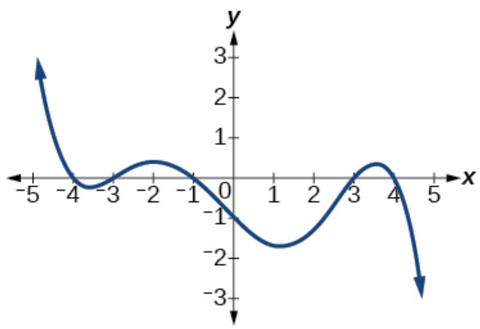


A polynomial of degree n will have at most $n-1$ turning points.

The least possible degree of the given polynomial function is 3 because it has 2 turning points.

32. -

33.

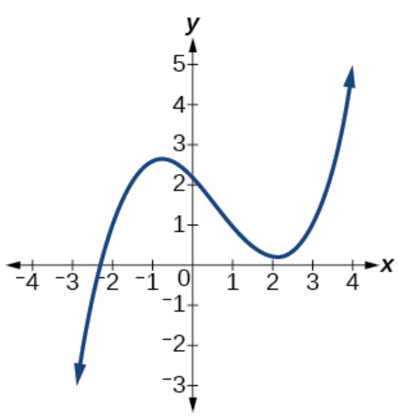


A polynomial of degree n will have at most $n-1$ turning points.

The least possible degree of the given polynomial function is 5 because it has 4 turning points.

34. –

35.

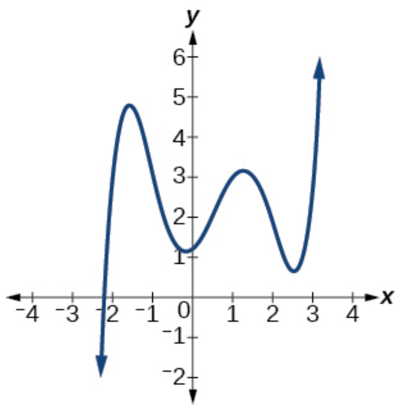


A polynomial of degree n will have at most $n-1$ turning points.

The least possible degree of the given polynomial function is 3 because it has 2 turning points.

36. -

37.



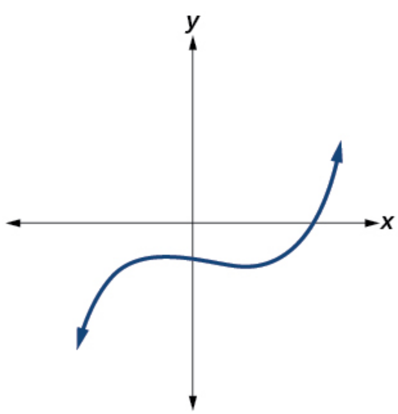
A polynomial of degree n will have at most $n-1$ turning points.

The least possible degree of the given polynomial function is 5 because it has 4 turning points.

38. -

For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

39.

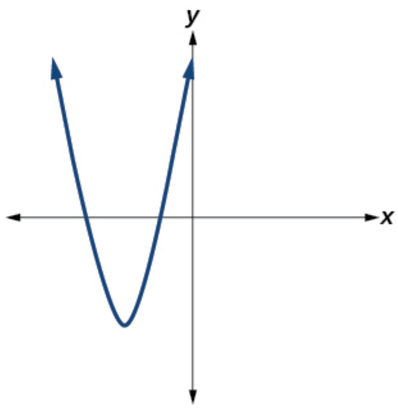


Yes; The graph of the function provided is a graph of a polynomial function because it is both continuous and smooth.

The graph has 1 x-intercept, suggesting a degree of 1 or greater, and 2 turning points, suggesting a degree of 3 or greater. Based on this, it would be reasonable to conclude that the degree is at least 3.

40. -

41.

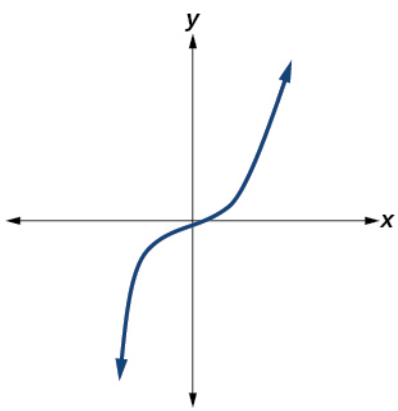


Yes; The graph of the function provided is a graph of a polynomial function because it is both continuous and smooth.

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 1 turning point, suggesting a degree of 2 or greater. Based on this, it would be reasonable to conclude that the degree is at least 2.

42. -

43.

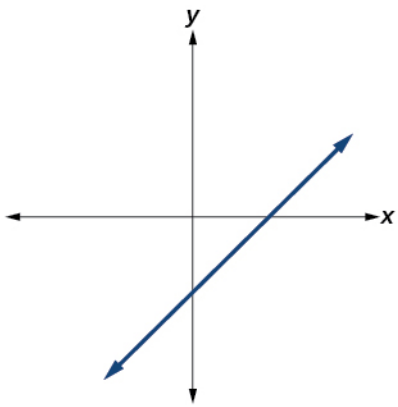


Yes; The graph of the function provided is a graph of a polynomial function because it is both continuous and smooth.

The graph has 1 x-intercept, suggesting a degree of 1 or greater, and 0 turning points, suggesting a degree of 1 or greater. Based on this, it would be reasonable to conclude that the degree is at least 1.

44. -

45.



Yes; The graph of the function provided is a graph of a polynomial function because it is a graph of a linear equation.

The graph has 1 x-intercept, suggesting a degree of 1 or greater, and 0 turning points, suggesting a degree of 1 or greater. Based on this, it would be reasonable to conclude that the degree is at least 1.

Numeric

For the following exercises, make a table to confirm the end behavior of the function.

46. -

47. $f(x) = x^4 - 5x^2$

Consider the following table.

x	$f(x)$
10	9,500
100	99,950,000
–10	9,500
–100	99,950,000

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ increase without bound. We can describe the end behavior symbolically by writing

as $x \to -\infty, f(x) \to \infty$

as $x \to \infty, f(x) \to \infty$.

48. -

49. $f(x) = (x-1)(x-2)(3-x)$

Consider the following table.

x	$f(x)$
10	–504
100	–941,094
–10	1,716
–100	1,061,106

As the input values x get very large, the output values $f(x)$ decrease without bound. As the input values x get very small, the output values $f(x)$ increase without bound. We can describe the end behavior symbolically by writing

as $x \to -\infty$, $f(x) \to \infty$

as $x \to \infty$, $f(x) \to -\infty$.

50. -

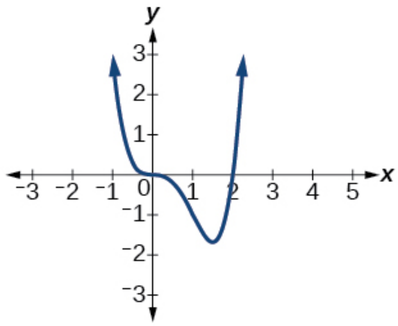
Technology

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

51. $f(x) = x^3(x-2)$

Press $\boxed{Y=}$ $\boxed{x}$ $\boxed{\wedge}$ $\boxed{3}$ $\boxed{*}$ $\boxed{(}$ $\boxed{x}\boxed{-}\boxed{2}$ $\boxed{)}$

This graphs the curve in a standard window, similar to the graph below.



To find the y-intercept, press $\boxed{\text{TRACE}}$ $\boxed{0}$ $\boxed{\text{ENTER}}$.

The y-intercept is $(0, 0)$.

To find x-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercept.

The x-intercepts are $(0, 0)$, $(2, 0)$.

The left and right boundaries to the graph indicate the end-behavior of the graph.

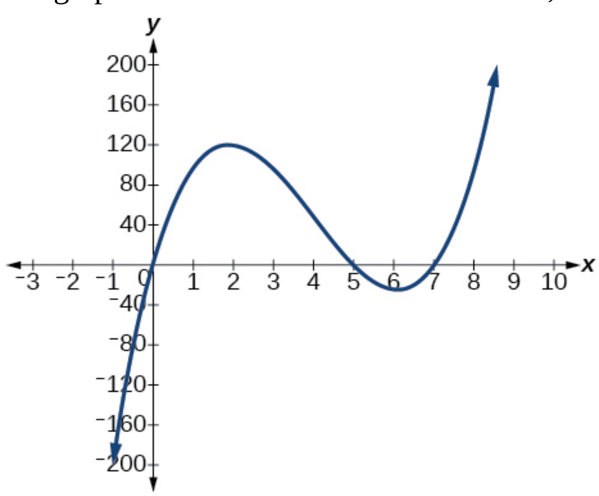
As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

52. -

53. $f(x) = x(14-2x)(10-2x)$

Press $\boxed{Y=}$ $\boxed{x}\boxed{(}\boxed{14}\boxed{-}\boxed{2}\boxed{x}\boxed{)}\boxed{(}\boxed{10}\boxed{-}\boxed{2}\boxed{x}\boxed{)}$

This graphs the curve in a standard window, similar to the graph below.



To find the y-intercept, press $\boxed{\text{TRACE}}$ $\boxed{0}$ $\boxed{\text{ENTER}}$.

The y-intercept is $(0, 0)$.

To find x-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercepts.

The x-intercepts are $(0, 0)$, $(5, 0)$, $(7, 0)$.

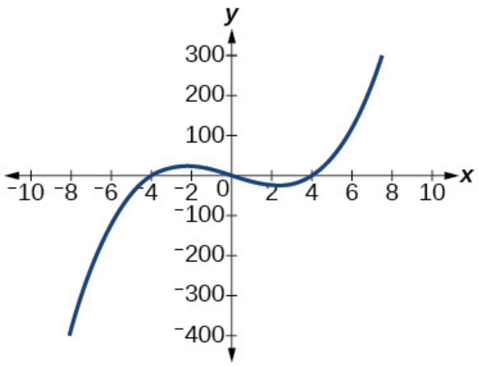
The left and right boundaries to the graph indicate the end-behavior of the graph.
As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

54. -

55. $f(x) = x^3 - 16x$

Press $\boxed{Y =}$ $\boxed{x}\boxed{\wedge}\boxed{3}\boxed{-}\boxed{16}\boxed{x}$

This graphs the curve in a standard window, similar to the graph below.



To find the y-intercept, press $\boxed{\text{TRACE}}$ $\boxed{0}$ $\boxed{\text{ENTER}}$.

The y-intercept is $(0, 0)$.

To find x-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercepts.

The x-intercepts are $(-4, 0)$, $(0, 0)$, $(4, 0)$.

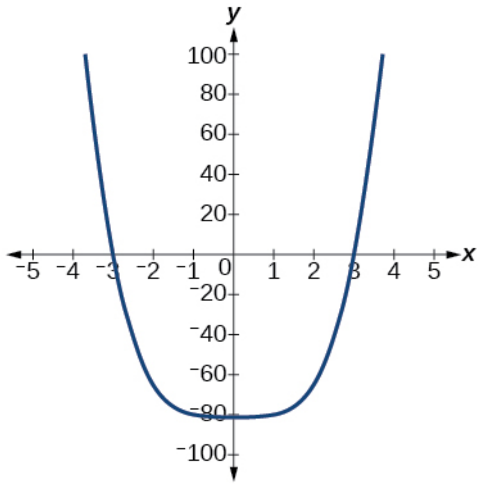
The left and right boundaries to the graph indicate the end-behavior of the graph.
As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

56. -

57. $f(x) = x^4 - 81$

Press $\boxed{Y=}$ $\boxed{x}\boxed{\wedge}\boxed{4}\boxed{-}\boxed{81}$

This graphs the curve in a standard window, similar to the graph below.



To find the y-intercept, press $\boxed{\text{TRACE}}$ $\boxed{0}$ $\boxed{\text{ENTER}}$.

The y-intercept is $(0, -81)$.

To find x-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercept.

The x-intercepts are $(3, 0)$, $(-3, 0)$.

The left and right boundaries to the graph indicate the end-behavior of the graph.

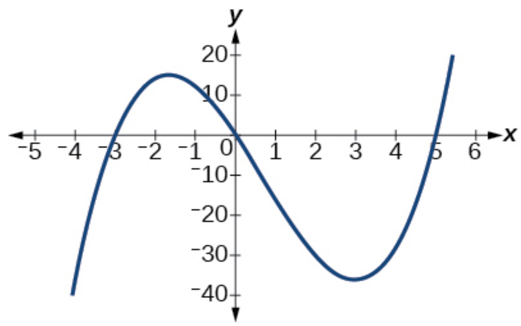
As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

58. -

59. $f(x) = x^3 - 2x^2 - 15x$

Press $\boxed{Y=}$ $\boxed{x}\boxed{\wedge}\boxed{3}\boxed{-}\boxed{2}\boxed{x}\boxed{\wedge}\boxed{2}\boxed{-}\boxed{15}\boxed{x}$

This graphs the curve in a standard window, similar to the graph below.



To find the y-intercept, press $\boxed{\text{TRACE}}$ $\boxed{0}$ $\boxed{\text{ENTER}}$.

The y-intercept is $(0,\ 0)$.

To find x-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercepts.

The x-intercepts are $(-3,\ 0)$, $(0,\ 0)$, $(5,\ 0)$.

The left and right boundaries to the graph indicate the end-behavior of the graph.

As $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

60. -

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

61. The y-intercept is $(0, -4)$. The x-intercepts are $(-2,\ 0)$, $(2,\ 0)$. Degree is 2.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

Remember that the degree of the given equation is 2.

First find the equation in x-intercept form:

$f(x) = a(x-b)(x-c)$ where a is some constant and b and c are x-intercepts.

Substitute the x-intercepts in the above equation.

$$\begin{aligned} f(x) &= a\left(x-(-2)\right)\left(x-2\right) \\ &= a\left(x+2\right)\left(x-2\right) \\ &= a\left(x^2-4\right) \end{aligned}$$

Substitute the y-intercept to find the constant a.

$$\begin{aligned} -4 &= a\left(0^2-4\right) \\ -4 &= -4a \\ a &= 1 \end{aligned}$$

Thus, the required equation is $f(x)=x^2-4$.

62. -

63. The y-intercept is $(0,0)$. The x-intercepts are $(0,\ 0)$, $(2,\ 0)$. Degree is 3.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$.

Since the graph has 2 x-intercepts and degree 3, one of the factors has multiplicity 2.
First find the equation in x-intercept form:

$f(x)=a\left(x-b\right)^2\left(x-c\right)$ or $f(x)=a\left(x-b\right)\left(x-c\right)^2$ where a is some constant and b and c are x-intercepts.

Substitute the x-intercepts in any of the equation.

$$\begin{aligned} f(x) &= a\left(x-0\right)\left(x-2\right)^2 \\ &= ax\left(x-2\right)^2 \end{aligned} \quad \text{or} \quad \begin{aligned} f(x) &= a\left(x-0\right)^2\left(x-2\right) \\ &= ax^2\left(x-2\right) \end{aligned}$$

From the end behavior of the graph, the leading coefficient (a) should be 1, not –1.
Thus, the required equation is

$$\begin{aligned} f(x) &= x\left(x-2\right)^2 \\ &= x\left(x^2-4x+4\right) \\ &= x^3-4x^2+4x \end{aligned} \quad \text{or} \quad \begin{aligned} f(x) &= x^2\left(x-2\right) \\ &= x^3-2x \end{aligned}$$

64. -

65. The y-intercept is $(0,\ 1)$. There is no x-intercept. Degree is 4.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

Since the graph has no x-intercept and has degree 4, the equation should be in the form $f(x)=ax^4+c$ where a and c are constants.

From the end behavior of the graph, the leading coefficient (a) should be 1, not –1.

$f(x)=x^4+c$

Substitute the y-intercept to find the constant c.

$1 = 0 + c$

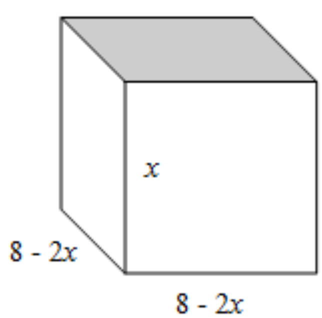
$c = 1$

Thus, the required equation is $f(x) = x^4 + 1$

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

66. -
67. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.
    The edge of the cube depends on the number of minutes m that have elapsed. This relationship is linear.
    $L(m) = 2m + 3$
    We can combine this with the formula for the volume V of a cube.
    $V(L) = L^3$
    Composing these functions gives a formula for the volume in terms of minutes.
    $$\begin{aligned} V(m) &= V(L(m)) \\ &= V(2m+3) \\ &= (2m+3)^3 \\ &= 8m^3 + 36m^2 + 54m + 27 \quad \text{Remember that } (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \end{aligned}$$
68. -
69. An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.
    Construct the open box as shown.



The open box has a length of $8 - 2x$ inches, a width of $8 - 2x$ inches and a height of x inches.

We can combine this with the formula for the volume V of a cube.

$V = \text{length} \cdot \text{width} \cdot \text{height}$

Composing these functions gives a formula for the volume in terms of x.

$V(x) = (8-2x)(8-2x)x$

$= \left(64 - 16x - 16x + 4x^2\right)x$ Multiply.

$= \left(64 - 32x + 4x^2\right)x$ Combine like terms.

$= 4x^3 - 32x^2 + 64x$ Multiply and rearrange the polynomial in descending order.

70. -

Chapter 5
Polynomial and Rational Functions
5.3 Graphs of Polynomial Functions

Verbal

1. What is the difference between an *x*-intercept and a zero of a polynomial function f ?

   The *x*-intercept is where the graph of the function crosses the *x*-axis, and the zero of the function is the input value for which $f(x)=0$.
2. -
3. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.
   If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b.
4. -
5. If the graph of a polynomial just touches the *x*-axis and then changes direction, what can we conclude about the factored form of the polynomial?
   There will be a factor raised to an even power.

Algebraic

For the following exercises, find the *x*- or *t*-intercepts of the polynomial functions.

6. -

7. $C(t)=3(t+2)(t-3)(t+5)$

   The *t*-intercepts can be found by solving $C(t)=0$.

   $3(t+2)(t-3)(t+5)=0$

   $t+2=0 \quad \text{or} \quad t-3=0 \quad \text{or} \quad t+5=0$

   $t=-2 \qquad\qquad t=3 \qquad\qquad t=-5$

   So the *t*-intercepts are $(-2,0)$, $(3,0)$, $(-5,0)$.
8. -
9. $C(t)=2t(t-3)(t+1)^2$

   The *t*-intercepts can be found by solving $C(t)=0$.

   $2t(t-3)(t+1)^2=0$

   $t=0 \quad \text{or} \quad t-3=0 \quad \text{or} \quad t+1=0$

   $t=3 \qquad\qquad t=-1$ (twice)

   So the *t*-intercepts are $(3,0)$, $(-1,0)$, $(0,0)$.

10. -

11. $C(t) = 4t^4 + 12t^3 - 40t^2$

Find solutions for $C(t) = 0$ by factoring.

$4t^4 + 12t^3 - 40t^2 = 0$

$4t^2(t^2 + 3t - 10) = 0$ Factor out the common factor.

$4t^2(t-2)(t+5) = 0$ Factor the trinomial.

Set each factor equal to zero.

$t = 0$ (twice) or $t - 2 = 0$ or $t + 5 = 0$

$t = 2$ $t = -5$

So the *t*-intercepts are $(0,0)$, $(-5,0)$, $(2,0)$.

12. -

13. $f(x) = x^3 + x^2 - 20x$

Find solutions for $f(x) = 0$ by factoring.

$x^3 + x^2 - 20x = 0$

$x(x^2 + x - 20) = 0$ Factor out the common factor.

$x(x-4)(x+5) = 0$ Factor the trinomial.

Set each factor equal to zero.

$x = 0$ or $x - 4 = 0$ or $x + 5 = 0$

$x = 4$ $x = -5$

So the *x*-intercepts are $(0,0)$, $(-5,0)$, $(4,0)$.

14. -

15. $f(x) = x^3 + x^2 - 4x - 4$

Find solutions for $f(x) = 0$ by factoring.

$x^3 + x^2 - 4x - 4 = 0$ Factor by grouping.

$x^2(x+1) - 4(x+1) = 0$ Factor out the common factor.

$(x^2 - 4)(x+1) = 0$ Factor the difference of squares.

$(x+2)(x-2)(x+1) = 0$ Set each factor equal to zero.

$x + 2 = 0$ or $x - 2 = 0$ or $x + 1 = 0$

$x = -2$ $x = 2$ $x = -1$

So the *x*-intercepts are $(-2,0)$, $(2,0)$, $(-1,0)$.

16. -

17. $f(x)=2x^3-x^2-8x+4$

Find solutions for $f(x)=0$ by factoring.

$2x^3-x^2-8x+4=0$ Factor by grouping.

$x^2(2x-1)-4(2x-1)=0$ Factor out the common factor.

$(x^2-4)(2x-1)=0$ Factor the difference of squares.

$(x+2)(x-2)(2x-1)=0$ Set each factor equal to zero.

$x+2=0$ or $x-2=0$ or $2x-1=0$

$x=-2$ $x=2$ $x=\frac{1}{2}$

So the x-intercepts are $(-2,0)$, $(2,0)$, $\left(\frac{1}{2},0\right)$.

18. -

19. $f(x)=2x^4+6x^2-8$

Find solutions for $f(x)=0$ by factoring.

$2x^4+6x^2-8=0$

$2u^2+6u-8=0$ Let $u=x^2$.

$2(u^2+3u-4)=0$ Factor out the common factor.

$2(u-1)(u+4)=0$ Factor the trinomial.

$2(x^2-1)(x^2+4)=0$ Let $x^2=u$.

$2(x-1)(x+1)(x^2+4)=0$ Factor the difference of squares.

Set each factor equal to zero and consider only the real roots.

$x-1=0$ or $x+1=0$

$x=1$ $x=-1$

So the x-intercepts are $(1,0)$, $(-1,0)$.

20. -

21. $f(x)=x^6-2x^4-3x^2$

Find solutions for $f(x)=0$ by factoring.

$x^6-2x^4-3x^2=0$

$x^2(x^4-2x^2-3)=0$ Factor out the common factor.

$x^2(x^2-3)(x^2+1)=0$ Factor the trinomial.

Set each factor equal to zero and consider only the real roots.

$$x^2-3=0 \quad \text{or} \quad x^2=0$$
$$x^2=3 \qquad\qquad x=0$$
$$x=\pm\sqrt{3}$$

So the *x*-intercepts are $(0,0)$, $(\sqrt{3},0)$, $(-\sqrt{3},0)$.

22. -

23. $f(x)=x^5-5x^3+4x$

Find solutions for $f(x)=0$ by factoring.

$$x^5-5x^3+4x=0$$
$$x\left(x^4-5x^2+4\right)=0 \quad \text{Factor out the common factor.}$$
$$x^2\left(x^2-1\right)\left(x^2-4\right)=0 \quad \text{Factor the trinomial.}$$

Set each factor equal to zero.

$$x^2=0 \quad \text{or} \quad x^2-1=0 \quad \text{or} \quad x^2-4=0$$
$$x=0 \qquad x^2=1 \qquad x^2=4$$
$$x=\pm 1 \qquad x=\pm 2$$

So the *x*-intercepts are $(0,0)$, $(1,0)$, $(-1,0)$, $(2,0)$, $(-2,0)$.

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

24. -

25. $f(x)=x^3-9x$, between $x=2$ and $x=4$.

Substitute $x=2$ and $x=4$ into the function $f(x)=x^3-9x$ and simplify.

$$f(2)=2^3-9(2)$$
$$=8-18$$
$$=-10$$
$$f(4)=4^3-9(4)$$
$$=64-36$$
$$=28$$

Because f is a polynomial function and since $f(2)$ is negative and $f(4)$ is positive, there is at least one real zero between $x=2$ and $x=4$.

26. -

27. $f(x) = -x^4 + 4,$ between $x = 1$ and $x = 3$.

Substitute $x = 1$ and $x = 3$ into the function $f(x) = -x^4 + 4$ and simplify.

$$\begin{aligned} f(1) &= -1^4 + 4 \\ &= -1 + 4 \\ &= 3 \\ f(3) &= -3^4 + 4 \\ &= -81 + 4 \\ &= -77 \end{aligned}$$

Because f is a polynomial function and since $f(1)$ is positive and $f(3)$ is negative, there is at least one real zero between $x = 1$ and $x = 3$.

28. -

29. $f(x) = x^3 - 100x + 2,$ between $x = 0.01$ and $x = 0.1$

Substitute $x = 0.01$ and $x = 0.1$ into the function $f(x) = x^3 - 100x + 2$ and simplify.

$$\begin{aligned} f(0.01) &= (0.01)^3 - 100(0.01) + 2 \\ &= 0.000001 - 1 + 2 \\ &= 1.000001 \\ f(0.1) &= (0.1)^3 - 100(0.1) + 2 \\ &= 0.001 - 10 + 2 \\ &= -7.999 \end{aligned}$$

Because f is a polynomial function and since $f(0.01)$ is positive and $f(0.1)$ is negative, there is at least one real zero between $x = 0.01$ and $x = 0.1$.

For the following exercises, find the zeros and give the multiplicity of each.

30. -

31. $f(x) = x^2(2x+3)^5(x-4)^2$

The zero associated with the factor, $x = 0,$ has multiplicity 2 because the factor x occurs twice.

The next zero associated with the factor, $x = -\frac{3}{2},$ has multiplicity 5 because the factor $(2x+3)$ occurs five times. The last zero associated with the factor, $x = 4,$ has multiplicity 2 because the factor $(x-4)$ occurs twice.

32. -

33. $f(x)=x^2\left(x^2+4x+4\right)$

Find zeros for $f(x)$ by factoring.

$$\begin{aligned} f(x) &= x^2\left(x^2+4x+4\right) \\ &= x^2\left(x+2\right)^2 \end{aligned}$$

The zero associated with the factor, $x=0,$ has multiplicity 2 because the factor x occurs twice. The next zero associated with the factor, $x=-2,$ has multiplicity 2 because the factor $\left(x+2\right)$ occurs twice.

34. -

35. $f(x)=\left(3x+2\right)^5\left(x^2-10x+25\right)$

Find zeros for $f(x)$ by factoring.

$$\begin{aligned} f(x) &= \left(3x+2\right)^5\left(x^2-10x+25\right) \\ &= \left(3x+2\right)^5\left(x-5\right)^2 \end{aligned}$$

The zero associated with the factor, $x=-\frac{2}{3},$ has multiplicity 5 because the factor $\left(3x+2\right)$ occurs five times. The next zero associated with the factor, $x=5,$ has multiplicity 2 because the factor $\left(x-5\right)$ occurs twice.

36. -

37. $f(x)=x^6-x^5-2x^4$

Find zeros for $f(x)$ by factoring.

$$\begin{aligned} f(x) &= x^6-x^5-2x^4 \\ &= x^4\left(x^2-x-2\right) \\ &= x^4\left(x-2\right)\left(x+1\right) \end{aligned}$$

The zero associated with the factor, $x=0,$ has multiplicity 4 because the factor x occurs four times. The next zero associated with the factor, $x=2,$ has multiplicity 1 because the factor $\left(x-2\right)$ only once. The last zero associated with the factor, $x=-1,$ has multiplicity 1 because the factor $\left(x+1\right)$ occurs only once.

38. -

39. $f(x)=4x^5-12x^4+9x^3$

Find zeros for $f(x)$ by factoring.

$$\begin{aligned} f(x) &= 4x^5-12x^4+9x^3 \\ &= x^3\left(4x^2-12x+9\right) \\ &= x^3\left(2x-3\right)^2 \end{aligned}$$

The zero associated with the factor, $x=0,$ has multiplicity 3 because the factor x occurs three times. The next zero associated with the factor, $x=\frac{3}{2},$ has multiplicity 2 because the factor $(2x-3)$ occurs twice.

40. -

41. $f(x)=4x^4\left(9x^4-12x^3+4x^2\right)$

Find zeros for $f(x)$ by factoring.

$$\begin{aligned} f(x) &= 4x^4\left(9x^4-12x^3+4x^2\right) \\ &= 4x^6\left(9x^2-12x+4\right) \\ &= 4x^6\left(3x-2\right)^2 \end{aligned}$$

The zero associated with the factor, $x=0,$ has multiplicity 6 because the factor x occurs six times. The next zero associated with the factor, $x=\frac{2}{3},$ has multiplicity 2 because the factor $(3x-2)$ occurs twice.

Graphical

For the following exercises, graph the polynomial functions. Note *x*- and *y*-intercepts, multiplicity, and end behavior.

42. -

43. $g(x)=(x+4)(x-1)^2$

The zero associated with the factor, $x=-4,$ has multiplicity 1 because the factor $x+4$ occurs only once.

The next zero associated with the factor, $x=1,$ has multiplicity 2 because the factor $x-1$ occurs twice.

So the *x*-intercepts are $(-4,0)$ with multiplicity 1, $(1,0)$ with multiplicity 2.

The *y*-intercept is found by evaluating $f(0).$

$f(0)=(0+4)(0-1)^2=4\cdot 1=4$

The *y*-intercept is $(0,4)$.

Since the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound.

We can describe the end behavior symbolically by writing

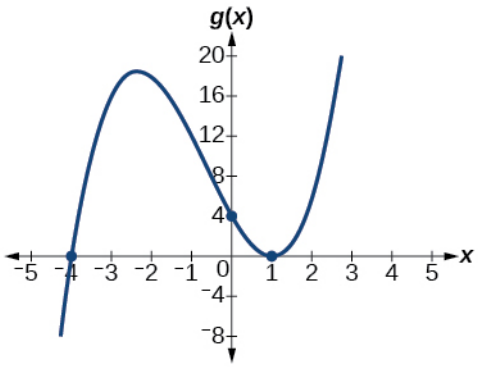
as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$.

To sketch this, we consider that:

As the first zero -4 has multiplicity 1, the graph looks almost linear at this point.

As the next zero 1 has multiplicity 2, the graph touches the *x*-axis at this point.

At $(0,4)$, the graph crosses the *y*-axis at the *y*-intercept.



44. -

45. $k(x)=(x-3)^3(x-2)^2$

The zero associated with the factor, $x=3$, has multiplicity 3 because the factor $x-3$ occurs three times.

The next zero associated with the factor, $x=2$, has multiplicity 2 because the factor $x-2$ occurs twice.

So the *x*-intercepts are $(3,0)$ with multiplicity 3, $(2,0)$ with multiplicity 2.

The *y*-intercept is found by evaluating $f(0)$.

$f(0)=(0-3)^3(0-2)^2=-27\cdot 4=-108$

The *y*-intercept is $(0,-108)$.

Since the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound.

We can describe the end behavior symbolically by writing

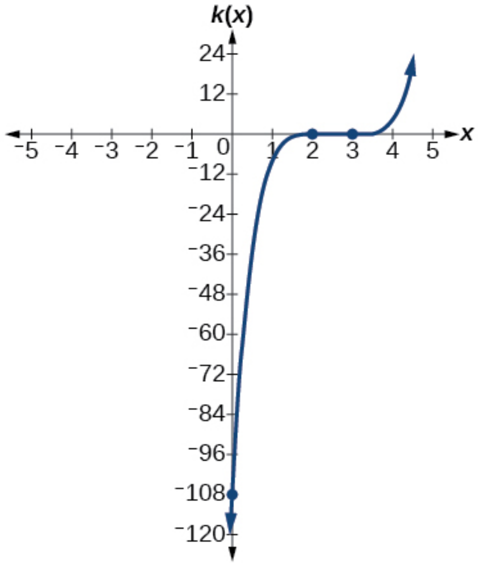
as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$.

To sketch this, we consider that:

As the first zero 3 has multiplicity 3, the graph crosses the *x*-axis at this point.

As the next zero 2 has multiplicity 2, the graph touches the *x*-axis at this point.

At $(0,-108)$, the graph crosses the *y*-axis at the *y*-intercept.



46. -

47. $n(x)=-3x(x+2)(x-4)$

The first zero associated with the factor, $x=0$, has multiplicity 1 because the factor x occurs only once.

The next zero associated with the factor, $x=-2$, has multiplicity 1 because the factor $x+2$ occurs only once.

The last zero associated with the factor, $x=4$, has multiplicity 1 because the factor $x-4$ occurs only once.

So the *x*-intercepts are $(0,\ 0)$, $(-2,\ 0)$, $(4,0)$ with multiplicity 1.

The *y*-intercept is found by evaluating $f(0)$.

$$f(0)=-3(0)(0+2)(0-4)=0$$

The *y*-intercept is $(0,0)$.

Since the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound.

We can describe the end behavior symbolically by writing

as $x\to-\infty$, $f(x)\to-\infty$, as $x\to\infty$, $f(x)\to\infty$.

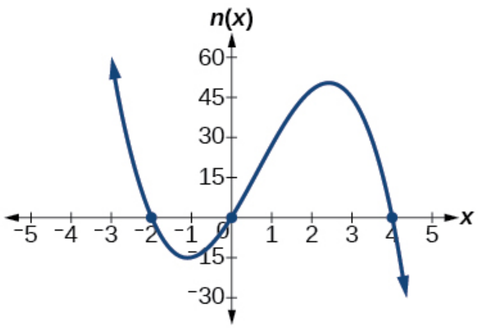
To sketch this, we consider that:

As the first zero 0 has multiplicity 1, the graph looks almost linear at this point.

As the next zero -2 has multiplicity 1, the graph looks almost linear at this point.

As the last zero 4 has multiplicity 1, the graph looks almost linear at this point.

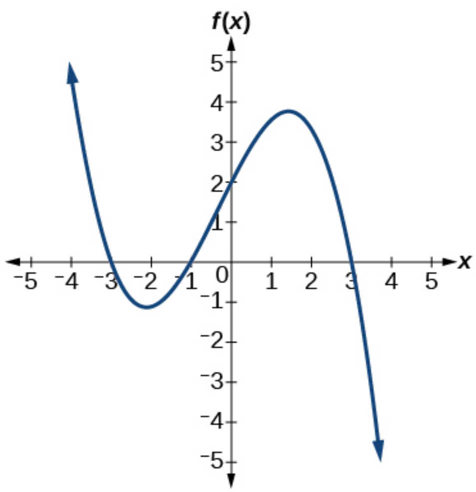
At $\left(0,0\right)$, the graph crosses the *y*-axis at the *y*-intercept.



For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

48. -

49.



Starting from the left, the first zero occurs at $x=-3$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The next zero occurs at $x=-1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x=3$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

Each *x*-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x)=a\left(x-3\right)\left(x+1\right)\left(x+3\right)$$

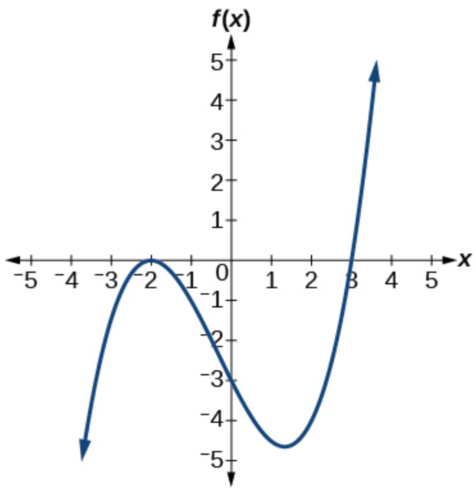
To determine the stretch factor, we utilize another point on the graph. We will use the *y*-intercept $\left(0,2\right)$, to solve for a.

$$
\begin{aligned}
f(0) &= a(0-3)(0+1)(0+3) \\
2 &= a(-3)(1)(3) \\
2 &= -9a \\
a &= -\frac{2}{9}
\end{aligned}
$$

The given information represents the function $f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$.

50. -

51.



Starting from the left, the first zero occurs at $x=-2$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of -2 has multiplicity 2.

The next zero occurs at $x=3$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a(x+2)^2(x-3)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0,-3)$, to solve for a.

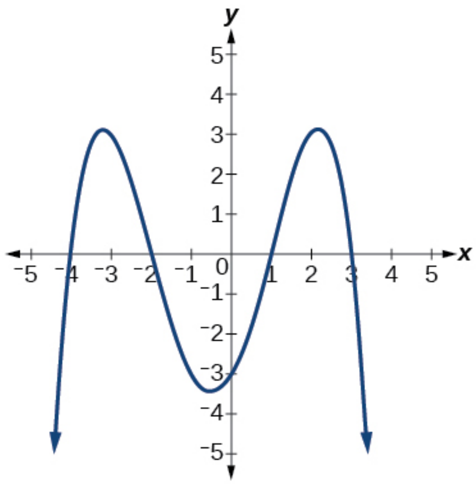
$$
\begin{aligned}
f(0) &= a(0+2)^2(0-3) \\
-3 &= a(4)(-3) \\
a &= \frac{1}{4}
\end{aligned}
$$

The given information represents the function $f(x) = \frac{1}{4}(x+2)^2(x-3)$.

52. -

For the following exercises, use the graph to identify zeros and multiplicity.

53.



Starting from the left, the first zero occurs at $x = -4.$ The graph looks almost linear at this point. This is a single zero of multiplicity 1.

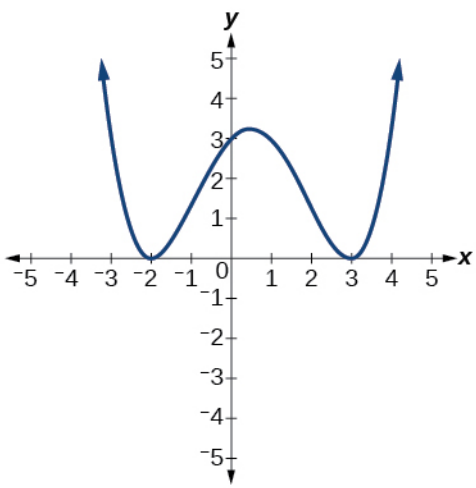
The next zero occurs at $x = -2.$ The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The next zero occurs at $x = 1.$ The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 3.$ The graph looks almost linear at this point. This is a single zero of multiplicity 1.

54. -

55.



Starting from the left, the first zero occurs at $x = -2.$ The graph touches the *x*-axis, so the multiplicity of the zero must be even. The zero of -2 has multiplicity $2.$

The next zero occurs at $x = 3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of 3 has multiplicity 2.

56. -

For the following exercises, use the given information about the polynomial graph to write the equation.

57. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. y-intercept at $(0, -4)$.

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a(x+2)(x-1)(x-3)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -4)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+2)(0-1)(0-3) \\ -4 &= a(0+2)(0-1)(0-3) \\ -4 &= 6a \\ a &= -\frac{2}{3} \end{aligned}$$

The given information represents the function $f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$.

58. -

59. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a(x-3)^2(x-1)^2(x+3)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, 9)$, to solve for a.

$$\begin{aligned} f(0) &= a(0-3)^2(0-1)^2(0+3) \\ 9 &= a(9)(1)(3) \\ 9 &= 27a \\ a &= \frac{1}{3} \end{aligned}$$

The given information represents the function $f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$.

60. -

61. Degree 5. Double zero at $x=1$, and triple zero at $x=3$. Passes through the point $(2,\ 15)$.

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a(x-1)^2(x-3)^3$$

To determine the stretch factor, we utilize another point on the graph. We will use the point $(2,\ 15)$, to solve for a.

$$f(2) = a(2-1)^2(2-3)^3$$
$$15 = a(1)(-1)$$
$$a = -1$$

The given information represents the function $f(x) = -15(x-1)^2(x-3)^3$.

62. -

63. Degree 3. Zeros at $x=-3$, $x=-2$ and $x=1$. y-intercept at $(0,\ 12)$.

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a(x+3)(x+2)(x-1)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, 12)$, to solve for a.

$$f(0) = a(0+3)(0+2)(0-1)$$
$$12 = a(3)(2)(-1)$$
$$12 = -6a$$
$$a = -2$$

The given information represents the function $f(x) = -2(x+3)(x+2)(x-1)$.

64. -

65. Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$ and roots of multiplicity 1 at $x=6$ and $x=-2$.

y-intercept at $(0,\ 18)$.

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a(2x-1)^2(x-6)(x+2)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0,18)$, to solve for a.

$$\begin{aligned} f(0) &= a(2(0)-1)^2(0-6)(0+2) \\ 18 &= a(1)(-6)(2) \\ 18 &= -12a \\ a &= -\frac{3}{2} \end{aligned}$$

The given information represents the function $f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$.

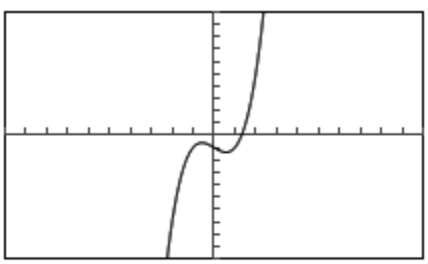
66. -

Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

67. $f(x) = x^3 - x - 1$

Set $\boxed{Y1=}$ $\boxed{x}$ $\boxed{\wedge}$ $\boxed{3}$ $\boxed{-}$ $\boxed{x}$ $\boxed{-}$ $\boxed{1}$



Use the calculator's Minimum feature to approximate the coordinates of the local minimum shown on the graph.

Open the Graph screen's CALCULATE menu by pressing $\boxed{2^{nd}}$ $\boxed{\text{CALC}}$.

Select 3: minimum.

The calculator returns to the graph and prompts you for a left bound.

Move the cursor until it is slightly to the left of the minimum. Press $\boxed{\text{ENTER}}$.

Move the cursor until it is slightly to the right of the minimum. Press $\boxed{\text{ENTER}}$.

The calculator prompts you for an initial guess for the minimum. Move the cursor close to the minimum. Press $\boxed{\text{ENTER}}$.

The calculator improves on the initial guess and returns an approximation of the coordinates of the local minimum point: $(.58, -1.38)$.

The Maximum feature of the CALCULATE menu can be used to find a local maximum using a similar procedure.

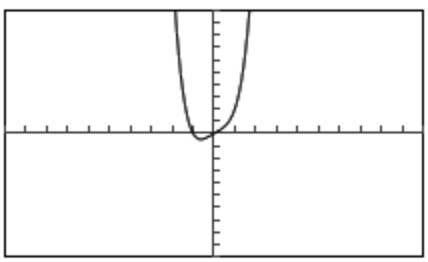
The local maximum point is $(-.58, -.62)$.

68. -

69. $f(x) = x^4 + x$

Since the given polynomial function is of even degree, it has a global minimum or maximum.

From the graph shown below, it appears that $f(x) = x^4 + x$ has a global minimum on [-1,0].



Enter the function in the Y= Editor.

Press $\boxed{\text{MATH}}$ $\boxed{6}$ to select fmin from Math menu.

Enter the function and then press $\boxed{,}$. If the function is stored in the Y=editor, press **VARS right arrow** and $\boxed{\text{ENTER}}$. Then press the number key corresponding to the number of the function in the Y= editor.

Enter the letter of the function variable (x) and press $\boxed{'}$.

Enter the lower limit of the interval containing the minimum, and press $\boxed{)'}$. Press $\boxed{\text{ENTER}}$ to find the absolute (global) minimum in that interval.

The global min is $(-.63, -.47)$.

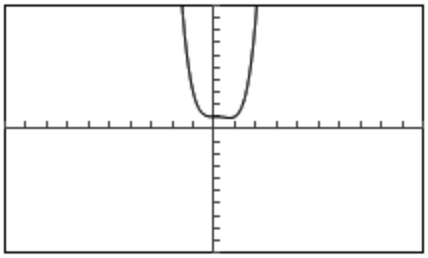
70. -

71. $f(x) = x^4 - x^3 + 1$

Since the given polynomial function is of even degree, it has a global minimum or maximum.

From the graph shown below, it appears that $f(x) = x^4 - x^3 + 1$ has a global minimum on

[0,1].



Enter the function in the Y= Editor.

Press $\boxed{\text{MATH}}$ $\boxed{6}$ to select fmin from Math menu.

Enter the function and then press $\boxed{,}$. If the function is stored in the Y=editor, press **VARS right arrow** and $\boxed{\text{ENTER}}$. Then press the number key corresponding to the number of the function in the Y= editor.

Enter the letter of the function variable (*x*) and press $\boxed{'}$.

Enter the lower limit of the interval containing the minimum, and press $\boxed{)'}$. Press $\boxed{\text{ENTER}}$ to find the absolute (global) minimum in that interval.

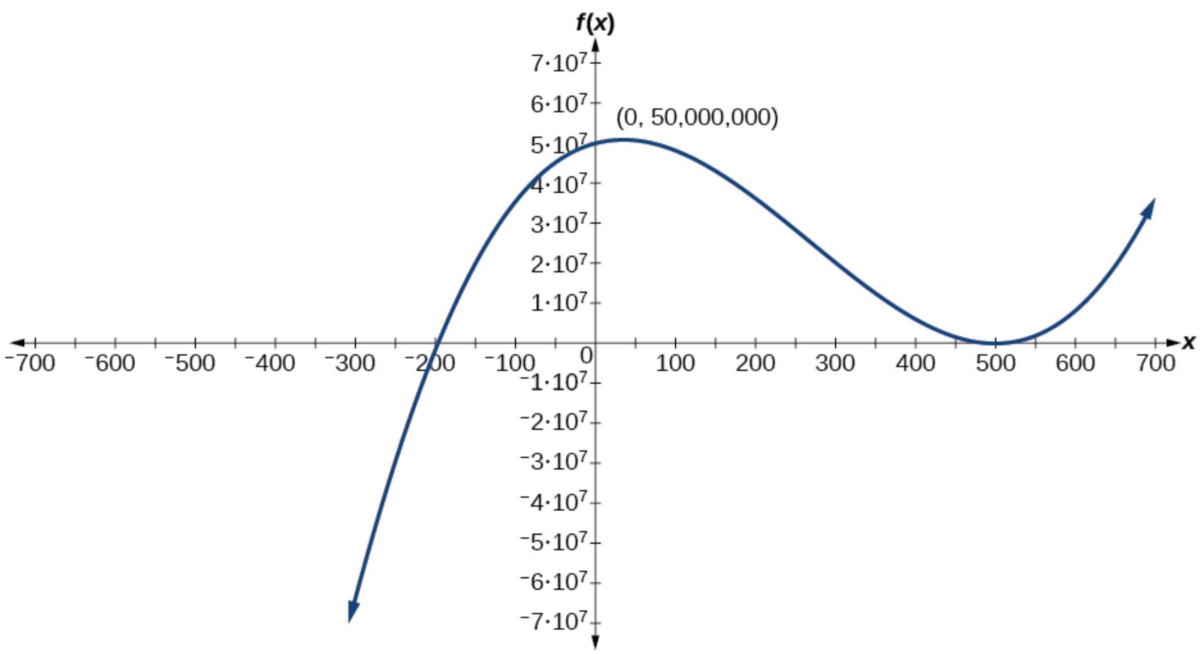
The global min is (.75, .89).

Extensions

For the following exercises, use the graphs to write a polynomial function of least degree.

72. –

73.



Starting from the left, the first zero occurs at $x = -200.$ The graph looks almost linear at this point. This is a zero of multiplicity 1.

The next zero occurs at $x = 500.$ The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of 500 has multiplicity $2.$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$f(x) = a\left(x - 500\right)^2\left(x + 200\right)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, 50000000)$, to solve for $a.$

$$\begin{aligned} f(0) &= a\left(0 - 500\right)^2\left(0 + 200\right) \\ 50000000 &= a(250000)(200) \\ 50000000 &= 50000000a \\ a &= 1 \end{aligned}$$

The given information represents the function $f(x) = (x - 500)^2(x + 200).$

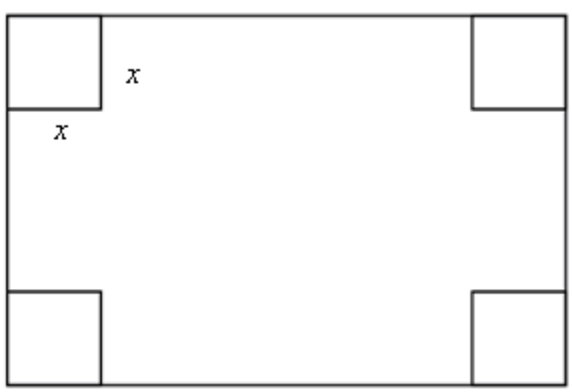
74. -

Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

75. A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of $x.$

We will start this problem by drawing a picture like below, labeling the width of the cut-out squares with a variable, $x.$



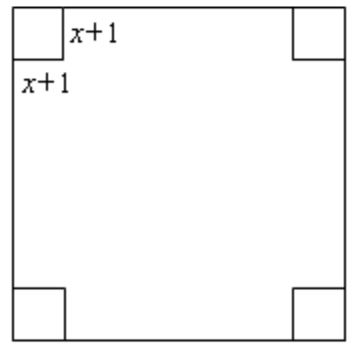
Notice that after a square is cut out from each end, it leaves a $(10 - 2x)$ cm by $(8 - 2x)$ cm rectangle for the base of the box, and the box will be x cm tall. This gives the volume

$$f(x) = (10-2x)(8-2x)x$$
$$= \left(80-36x+4x^2\right)x$$
$$= 4x^3-36x^2+80x$$

76. -

77. A square has sides of 12 units. Squares $x+1$ by $x+1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

We will start this problem by drawing a picture like below, labeling the width of the cut-out squares with a variable, $x+1$.



Notice that after a square is cut out from each end, it leaves a $\left(12-2(x+1)\right)=10-2x$ cm by $\left(12-2(x+1)\right)=10-x$ cm rectangle for the base of the box, and the box will be $x+1$cm tall. This gives the volume

$$f(x) = (10-2x)(10-2x)\left(x+1\right)$$
$$= \left(100-40x+4x^2\right)\left(x+1\right)$$
$$= 4x^3-36x^2+60x+100$$

78. -

79. A right circular cone has a radius of $3x+6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V=\frac{1}{3}\pi r^2 h$ for radius r and height h.

The given right circular cone has a radius of $3x+6$ and a height $3x+3$ units.

This gives the volume

$$f(x) = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(3x+6)^2\left(3x+3\right)$$
$$= \pi\left(9x^2+36x+36\right)\left(x+1\right)$$
$$= \pi\left(9x^3+45x^2+72x+36\right)$$

Chapter 5
Polynomial and Rational Functions
5.4 Dividing Polynomials

Verbal

1. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?
   The binomial is a factor of the polynomial.
2. -

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

3. $\left(x^2+5x-1\right)\div\left(x-1\right)$

$$x-1\overline{\big)\,x^2+5x-1}$$
Set up division problem.

$$\begin{array}{r} x \\ x-1\overline{\big)\,x^2+5x-1} \end{array}$$
x^2 divided by x is x.

$$\begin{array}{rl} x & \\ x-1\overline{\big)\,x^2+5x-1} & \text{Multiply } x-1 \text{ by } x. \\ \underline{-(x^2-x)} & \text{Subtract.} \\ 6x-1 & \text{Bring down the next term.} \end{array}$$

$$\begin{array}{rl} x+6 & \\ x-1\overline{\big)\,x^2+5x-1} & 6x \text{ divided by } x \text{ is } 6. \\ \underline{-(x^2-x)} & \\ 6x-1 & \\ \underline{-(6x-6)} & \text{Multiply } x-1 \text{ by } 6. \\ 5 & \text{Subtract.} \end{array}$$

The quotient is $x+6$. The remainder is 5. We write the result as

$$\left(x^2+5x-1\right)\div\left(x-1\right)=x+6+\frac{5}{x-1}$$

4. -

5. $\left(3x^2+23x+14\right)\div\left(x+7\right)$

$$x+7\overline{\big)3x^2+23x+14}$$ Set up division problem.

$$\begin{array}{r} 3x \\ x+7\overline{\big)3x^2+23x+14} \end{array}$$ $3x^2$ divided by x is $3x$.

$$\begin{array}{r} 3x \\ x+7\overline{\big)3x^2+23x+14} \\ \underline{-(3x^2+21x)} \\ 2x+14 \end{array}$$

Multiply $x+7$ by $3x$.

Subtract.

Bring down the next term.

$$\begin{array}{r} 3x+2 \\ x+7\overline{\big)3x^2+23x+14} \\ \underline{-(3x^2+21)} \\ 2x+14 \\ \underline{-(2x+14)} \\ 0 \end{array}$$

$2x$ divided by x is 2.

Multiply $x+7$ by 2.

Subtract.

The quotient is $3x+2$. The remainder is 0. We write the result as $\left(3x^2+23x+14\right)\div\left(x+7\right)=3x+2$

6. -

7. $\left(6x^2-25x-25\right)\div\left(6x+5\right)$

$$6x+5\overline{\big)6x^2-25x-25}$$ Set up division problem.

$$\begin{array}{r} x \\ 6x+5\overline{\big)6x^2-25x-25} \end{array}$$ $6x^2$ divided by $6x$ is x.

$$6x+5\overline{\big)6x^2-25x-25}^{\;x}$$ Multiply $6x+5$ by x.

$\underline{-(6x^2+5x)}$ Subtract.

$-30x-25$ Bring down the next term.

$$6x+5\overline{\big)6x^2-25x-25}^{\;x-5}$$ $-30x$ divided by $6x$ is -5.

$\underline{-(6x^2+5x)}$

$-30x-25$

$\underline{-(-30x-25)}$ Multiply $6x+5$ by -5.

0 Subtract.

The quotient is $x-5$. The remainder is 0. We write the result as

$\left(6x^2-25x-25\right)\div\left(6x+5\right)=x-5$

8. -

9. $\left(2x^2-3x+2\right)\div\left(x+2\right)$

$x+2\overline{\smash{)}2x^2-3x+2}$ Set up division problem.

$$\begin{array}{r}2x\\x+2\overline{\smash{)}2x^2-3x+2}\end{array}$$

$2x^2$ divided by x is $2x$.

$$\begin{array}{rl}2x & \\ x+2\overline{\smash{)}2x^2-3x+2} & \text{Multiply } x+2 \text{ by } 2x.\\ \underline{-(2x^2+4x)} & \text{Subtract.}\\ -7x+2 & \text{Bring down the next term.}\end{array}$$

$$\begin{array}{rl}2x-7 & \\ x+2\overline{\smash{)}2x^2-3x+2} & -7x \text{ divided by } x \text{ is } -7.\\ \underline{-(2x^2+4x)} & \\ -7x+2 & \\ \underline{-(-7x-14)} & \text{Multiply } x+2 \text{ by } -7.\\ 16 & \text{Subtract.}\end{array}$$

The quotient is $2x-7$. The remainder is 16. We write the result as

$\left(2x^2-3x+2\right)\div\left(x+2\right)=2x-7+\dfrac{16}{x+2}$

10. -

11. $\left(3x^2-5x+4\right)\div\left(3x+1\right)$

$$3x+1\overline{\big)3x^2-5x+4}$$ Set up division problem.

$$\begin{array}{r} x \\ 3x+1\overline{\big)3x^2-5x+4} \end{array}$$ $3x^2$ divided by $3x$ is x.

$$\begin{array}{rl} x & \\ 3x+1\overline{\big)3x^2-5x+4} & \text{Multiply } 3x+1 \text{ by } x. \\ \underline{-(3x^2+x)} & \text{Subtract.} \\ -6x+4 & \text{Bring down the next term.} \end{array}$$

$$\begin{array}{rl} x-2 & \\ 3x+1\overline{\big)3x^2-5x+4} & -6x \text{ divided by } 3x \text{ is } -2. \\ \underline{-(3x^2+x)} & \\ -6x+4 & \\ \underline{-(-6x-2)} & \text{Multiply } 3x+1 \text{ by } -2. \\ 6 & \text{Subtract.} \end{array}$$

The quotient is $x-2.$ The remainder is $6.$ We write the result as

$$\left(3x^2-5x+4\right)\div\left(3x+1\right)=x-2+\frac{6}{3x+1}$$

12. -

13. $\left(2x^3+3x^2-4x+15\right)\div\left(x+3\right)$

$$
\begin{array}{r}
2x^2-3x+5 \\
x+3\overline{\smash{)}2x^3+3x^2-4x+15} \\
\underline{-(2x^3+6x^2)} \\
-3x^2-4x \\
\underline{-(-3x^2-9x)} \\
5x+15 \\
\underline{-(5x+15)} \\
0
\end{array}
$$

$2x^3$ divided by x is $2x^2$.

Multiply $x+3$ by $2x^2$.

Subtract. Bring down the next term. $-3x^2$ divided by x is $-3x$.

Multiply $x+3$ by $-3x$.

Subtract. Bring down the next term. $5x$ divided by x is 5.

Multiply $x+3$ by 5.

Subtract.

The quotient is $2x^2-3x+5$. The remainder is 0. We write the result as
$\left(2x^3+3x^2-4x+15\right)\div\left(x+3\right)=2x^2-3x+5$

For the following exercises, use synthetic division to find the quotient.

14. -
15. $\left(2x^3-6x^2-7x+6\right)\div(x-4)$

The binomial divisor is $x-4$ so $k=4$.

Add each column, multiply the result by 4, and repeat until the last column is reached.

$$
\begin{array}{r|rrrr}
4 & 2 & -6 & -7 & 6 \\
 & & 8 & 8 & 4 \\
\hline
 & 2 & 2 & 1 & 10
\end{array}
$$

The result is $2x^2+2x+1$. The remainder is 10.

Thus, $\left(2x^3-6x^2-7x+6\right)\div(x-4)=2x^2+2x+1+\dfrac{10}{x-4}$

16. -
17. $\left(4x^3-12x^2-5x-1\right)\div(2x+1)$

First, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{2x^3-6x^2-\frac{5}{2}x-\frac{1}{2}}{x+\frac{1}{2}}$$

Now, the binomial divisor is $x+\frac{1}{2}$ so $k=-\frac{1}{2}$.

Add each column, multiply the result by $-\frac{1}{2}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -6 & -\frac{5}{2} & -\frac{1}{2} \\ & & -1 & \frac{7}{2} & -\frac{1}{2} \\ \hline & 2 & -7 & 1 & -1 \end{array}$$

Thus, we find $\dfrac{2x^3-6x^2-\frac{5}{2}x-\frac{1}{2}}{x+\frac{1}{2}}=2x^2-7x+1-\dfrac{1}{x+\frac{1}{2}}$

so $\dfrac{\left(4x^3-12x^2-5x-1\right)}{(2x+1)}=2x^2-7x+1-\dfrac{2}{2x+1}$

18. -

19. $\left(3x^3-2x^2+x-4\right)\div\left(x+3\right)$

The binomial divisor is $x+3$ so $k=-3$.

Add each column, multiply the result by –3, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -3 & 3 & -2 & 1 & -4 \\ & & -9 & 33 & -102 \\ \hline & 3 & -11 & 34 & -106 \end{array}$$

The result is $3x^2-11x+34$. The remainder is -106.

Thus, $\left(3x^3-2x^2+x-4\right)\div\left(x+3\right)=3x^2-11x+34-\dfrac{106}{x+3}$

20. -

21. $\left(2x^3+7x^2-13x-3\right)\div\left(2x-3\right)$

First, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{x^3+\frac{7}{2}x^2-\frac{13}{2}x-\frac{3}{2}}{x-\frac{3}{2}}$$

Now, the binomial divisor is $x-\frac{3}{2}$so $k=\frac{3}{2}$.

Add each column, multiply the result by $\frac{3}{2}$, and repeat until the last column is reached.

$$\begin{array}{c|cccc} \frac{3}{2} & 1 & \frac{7}{2} & -\frac{13}{2} & -\frac{3}{2} \\ & & \frac{3}{2} & \frac{15}{2} & \frac{3}{2} \\ \hline & 1 & 5 & 1 & 0 \end{array}$$

Thus, we find $\dfrac{-3x^3+\frac{1}{2}x^2-2}{x-\frac{3}{2}}=x^2+5x+1$

so $\dfrac{\left(2x^3+7x^2-13x-3\right)}{\left(2x-3\right)}=x^2+5x+1$

22. -

23. $\left(4x^3-5x^2+13\right)\div(x+4)$

The binomial divisor is $x+4$ so $k=-4$.

Notice that there is no x-term. We will use a zero as the coefficient for that term.

Add each column, multiply the result by –4, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -4 & 4 & -5 & 0 & 13 \\ & & -16 & 84 & -336 \\ \hline & 4 & -21 & 84 & -323 \end{array}$$

The result is $4x^2-21x+84.$ The remainder is -323.

Thus, $\left(4x^3-5x^2+13\right)\div(x+4)=4x^2-21x+84-\dfrac{323}{x+4}$

24. -

25. $\left(x^3-21x^2+147x-343\right)\div\left(x-7\right)$

The binomial divisor is $x-7$ so $k=7$.

Add each column, multiply the result by 7, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} 7 & 1 & -21 & 147 & -343 \\ & & 7 & -98 & -343 \\ \hline & 1 & -14 & 49 & 0 \end{array}$$

The result is $x^2-14x+49.$ The remainder is 0.

Thus, $\left(x^3-21x^2+147x-343\right)\div\left(x-7\right)=x^2-14x+49$

26. -

27. $\left(9x^3-x+2\right)\div\left(3x-1\right)$

First, we need to change the coefficient of *x* in the divisor 1. Hence, we divide the numerator and denominator by 3 to get

$$\frac{3x^3-\frac{1}{3}x+\frac{2}{3}}{x-\frac{1}{3}}$$

Now, the binomial divisor is $x-\dfrac{1}{3}$ so $k=\dfrac{1}{3}$.

Notice that there is no *x*-term. We will use a zero as the coefficient for that term.

Add each column, multiply the result by $\dfrac{1}{3}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 0 & -\frac{1}{3} & \frac{2}{3} \\ & & 1 & \frac{1}{3} & 0 \\ \hline & 3 & 1 & 0 & \frac{2}{3} \end{array}$$

Thus, we find $\dfrac{3x^3-\frac{1}{3}x+\frac{2}{3}}{x-\frac{1}{3}}=3x^2+x+\dfrac{2/3}{\left(x-\frac{1}{3}\right)}$

so $\dfrac{\left(9x^3-x+2\right)}{(3x-1)}=3x^2+x+\dfrac{2}{3x-1}$

28. -

29. $\left(x^4+x^3-3x^2-2x+1\right)\div\left(x+1\right)$

The binomial divisor is $x+1$ so $k=-1$.

Add each column, multiply the result by –1, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -2 & 1 \\ & & -1 & 0 & 3 & -1 \\ \hline & 1 & 0 & -3 & 1 & 0 \end{array}$$

The result is x^3-3x+1. The remainder is 0.

Thus, $\left(x^4+x^3-3x^2-2x+1\right)\div\left(x+1\right)=x^3-3x+1$

30. -

31. $\left(x^4+2x^3-3x^2+2x+6\right)\div\left(x+3\right)$

The binomial divisor is $x+3$ so $k=-3$.

Add each column, multiply the result by –3, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} -3 & 1 & 2 & -3 & 2 & 6 \\ & & -3 & 3 & 0 & -6 \\ \hline & 1 & -1 & 0 & 2 & 0 \end{array}$$

The result is x^3-x^2+2. The remainder is 0.

Thus, $\left(x^4+2x^3-3x^2+2x+6\right)\div\left(x+3\right)=x^3-x^2+2$

32. -

33. $\left(x^4-8x^3+24x^2-32x+16\right)\div\left(x-2\right)$

The binomial divisor is $x-2$ so $k=2$.

Add each column, multiply the result by 2, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} 2 & 1 & -8 & 24 & -32 & 16 \\ & & 2 & -12 & 24 & -16 \\ \hline & 1 & -6 & 12 & -8 & 0 \end{array}$$

The result is $x^3-6x^2+12x-8$. The remainder is 0.

Thus, $\left(x^4-8x^3+24x^2-32x+16\right)\div\left(x-2\right)=x^3-6x^2+12x-8$

34. -

35. $\left(x^4-12x^3+54x^2-108x+81\right)\div\left(x-3\right)$

The binomial divisor is $x-3$ so $k=3$.

Add each column, multiply the result by 3, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} 3 & 1 & -12 & 54 & -108 & 81 \\ & & 3 & -27 & 81 & -81 \\ \hline & 1 & -9 & 27 & -27 & 0 \end{array}$$

The result is $x^3-9x^2+27x-27$. The remainder is 0.

Thus, $\left(x^4-12x^3+54x^2-108x+81\right)\div\left(x-3\right)=x^3-9x^2+27x-27$

36. -

37. $\left(4x^4+2x^3-4x^2+2x+2\right)\div\left(2x+1\right)$

First, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{2x^4+x^3-2x^2+x+1}{x+\frac{1}{2}}$$

Now, the binomial divisor is $x+\frac{1}{2}$ so $k=-\frac{1}{2}$.

Add each column, multiply the result by $-\frac{1}{2}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 2 & 1 & -2 & 1 & 1 \\ & & -1 & 0 & 1 & -1 \\ \hline & 2 & 0 & -2 & 2 & 0 \end{array}$$

Thus, we find $\dfrac{2x^4+x^3-2x^2+x+1}{x+\frac{1}{2}}=2x^3-2x+2$

so $\dfrac{\left(4x^4+2x^3-4x^2+2x+2\right)}{(2x+1)}=2x^3-2x+2$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

38. -

39. $x-2,\ 3x^4-6x^3-5x+10$

The binomial divisor is $x-2$ so $k=2$.

Notice that there is no x^2-term. We will use a zero as the coefficient for that term.

Add each column, multiply the result by 2, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} 2 & 3 & -6 & 0 & -5 & 10 \\ & & 6 & 0 & 0 & -10 \\ \hline & 3 & 0 & 0 & -5 & 0 \end{array}$$

The quotient is $3x^3-5$. The remainder is 0.

Thus, $x-2$ is a factor of $3x^4-6x^3-5x+10$

We write the result as

$$\frac{3x^4-6x^3-5x+10}{\left(x-2\right)}=(3x^3-5)$$

or

$$3x^4-6x^3-5x+10=\left(x-2\right)(3x^3-5)$$

40. -

41. $x-2,\ 4x^4-15x^2-4$

The binomial divisor is $x-2$ so $k=2$.

Notice that there are no x^3-term and x-term. We will use a zero as the coefficient for these terms.

Add each column, multiply the result by 2, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} 2 & 4 & 0 & -15 & 0 & -4 \\ & & 8 & 16 & 2 & 4 \\ \hline & 4 & 8 & 1 & 2 & 0 \end{array}$$

The quotient is $4x^3+8x^2+x+2$. The remainder is 0 .

Thus, $x-2$ is a factor of $4x^4-15x^2-4$

We write the result as

$$\frac{4x^4-15x^2-4}{(x-2)}=4x^3+8x^2+x+2$$

or

$$4x^4-15x^2-4=(x-2)(4x^3+8x^2+x+2)$$

42. -

43. $x+\frac{1}{3}, 3x^4+x^3-3x+1$

The binomial divisor is $x+\frac{1}{3}$ so $k=-\frac{1}{3}$.

Notice that there is no x^2 -term. We will use a zero as the coefficient for that term.

Add each column, multiply the result by $-\frac{1}{3}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 3 & 1 & 0 & -3 & 1 \\ & & -1 & 0 & 0 & 1 \\ \hline & 3 & 0 & 0 & -3 & 2 \end{array}$$

The quotient is $3x^3-3$. The remainder is 2 .

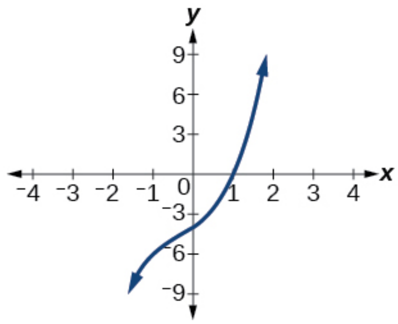
Thus, $x+\frac{1}{3}$ is not a factor of $3x^4+x^3-3x+1$

Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

44. -

45. Factor is (x^2+2x+4)

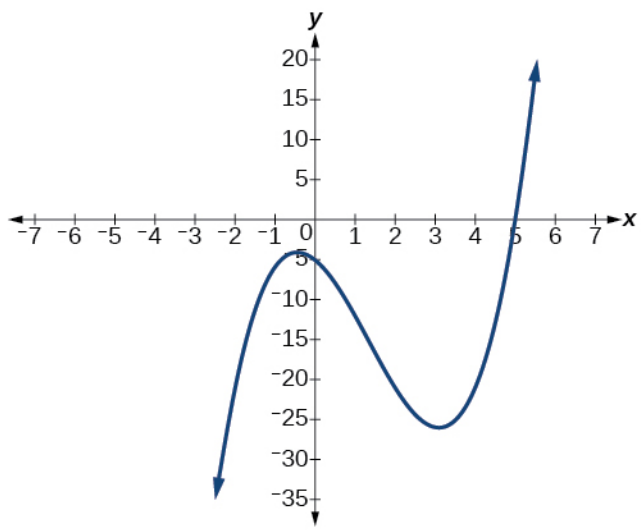


The graph shows a zero at $x = k = 1$. This confirms that $x - 1$ is a factor of the graph. Multiply the factors to get the third-degree polynomial.

$f(x) = (x-1)(x^2 + 2x + 4)$

46. -

47. Factor is $x^2 + x + 1$



The graph shows a zero at $x = k = 5$. This confirms that $x - 5$ is a factor of the graph. Multiply the factors to get the third-degree polynomial.

$f(x) = (x-5)(x^2 + x + 1)$

48. -

For the following exercises, use synthetic division to find the quotient and remainder.

49. $\dfrac{4x^3 - 33}{x-2}$

The binomial divisor is $x-2$ so $k = 2$.

Notice that there are no x^2 -term and x -term. We will use a zero as the coefficient for these terms.

Add each column, multiply the result by 2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} 2 & 4 & 0 & 0 & -33 \\ & & 8 & 16 & 32 \\ \hline & 4 & 8 & 16 & -1 \end{array}$$

The quotient is $4x^2 + 8x + 16$. The remainder is -1.

50. -

51. $\dfrac{3x^3 + 2x - 5}{x - 1}$

The binomial divisor is $x - 1$ so $k = 1$.

Notice that there is no x^2 -term. We will use a zero as the coefficient for that term.

Add each column, multiply the result by 1, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} 1 & 3 & 0 & 2 & -5 \\ & & 3 & 3 & 5 \\ \hline & 3 & 3 & 5 & 0 \end{array}$$

The quotient is $3x^2 + 3x + 5$. The remainder is 0.

52. -

53. $\dfrac{x^4 - 22}{x + 2}$

The binomial divisor is $x + 2$ so $k = -2$.

Notice that there are no x^3 -term, x^2 -term and x -term. We will use a zero as the coefficient for these terms.

Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & 0 & 0 & -22 \\ & & -2 & 4 & -8 & 16 \\ \hline & 1 & -2 & 4 & -8 & -6 \end{array}$$

The quotient is $x^3 - 2x^2 + 4x - 8$. The remainder is -6.

Technology

For the following exercises, use a calculator with CAS to answer the questions.

54. -

55. Consider $\frac{x^k+1}{x+1}$ for $k=1, 3, 5.$ What do you expect the result to be if $k=7$?

Use the Casio ClassPad.

Enter and highlight $\frac{x^7+1}{x+1}$ and tap

Interactive > Transformation > propFrac.

The output of the calculator is $x^6-x^5+x^4-x^3+x^2-x+1.$

56. -

57. Consider $\frac{x^k}{x+1}$ with $k=1, 2, 3.$ What do you expect the result to be if $k=4$?

Use the Casio ClassPad.

Enter and highlight $\frac{x^4}{x+1}$ and tap

Interactive > Transformation > propFrac.

The output of the calculator is $x^3-x^2+x-1+\frac{1}{x+1}.$

58. -

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

59. $\frac{x+1}{x-i}$

The binomial divisor is $x-i$ so $k=i.$

Add each column, multiply the result by i, and repeat until the last column is reached.

$$
\begin{array}{c|cc}
i & 1 & 1 \\
 & & i \\
\hline
 & 1 & 1+i
\end{array}
$$

The quotient is $1.$ The remainder is $1+i$. We write the result as

$$\frac{x+1}{x-i}=1+\frac{1+i}{x-i}$$

60. -

61. $\dfrac{x+1}{x+i}$

The binomial divisor is $x+i$ so $k=-i$.

Add each column, multiply the result by $-i$, and repeat until the last column is reached.

$$\begin{array}{r|rr} -i & 1 & 1 \\ & & -i \\ \hline & 1 & 1-i \end{array}$$

The quotient is 1. The remainder is $1-i$. We write the result as

$$\frac{x+1}{x+i}=1+\frac{1-i}{x+i}$$

62. -

63. $\dfrac{x^3+1}{x-i}$

The binomial divisor is $x-i$ so $k=i$.

Notice that there are no x^2-term and x-term. We will use a zero as the coefficient for these terms.

$$\begin{array}{r|rrrr} i & 1 & 0 & 0 & 1 \\ & & i & -1 & -i \\ \hline & 1 & i & -1 & 1-i \end{array}$$

The quotient is x^2+ix-1. The remainder is $1-i$. We write the result as

$$\frac{x^3+1}{x-i}=x^2+ix-1+\frac{1-i}{x-i}$$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

64. -

65. Length is $2x + 5$, area is $4x^3+10x^2+6x+15$

First write an equation by substituting the known values into the formula for the area of a rectangle.

$$A = l \cdot b$$

$$4x^3 + 10x^2 + 6x + 15 = (2x+5) \cdot b$$

Now solve for b using synthetic division.

$$b = \frac{4x^3 + 10x^2 + 6x + 15}{2x+5}$$

Now, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{2x^3 + 5x^2 + 3x + \frac{15}{2}}{x + \frac{5}{2}}$$

Now, the binomial divisor is $x + \frac{5}{2}$ so $k = -\frac{5}{2}$.

Add each column, multiply the result by $-\frac{5}{2}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -\frac{5}{2} & 2 & 5 & 3 & \frac{15}{2} \\ & & -5 & 0 & -\frac{15}{2} \\ \hline & 2 & 0 & 3 & 0 \end{array}$$

The quotient is $2x^2 + 3$ and the remainder is 0.

Thus, we find $\dfrac{2x^3 + 5x^2 + 3x + \frac{15}{2}}{x + \frac{5}{2}} = 2x^2 + 3$

so $\dfrac{4x^3 + 10x^2 + 6x + 15}{2x+5} = 2x^2 + 3$

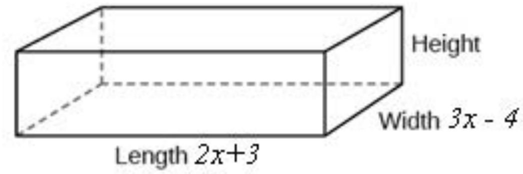
The width of the rectangle is $2x^2 + 3$.

66. -

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

67. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

There are a few ways to approach this problem. We need to divide the expression for the volume of the box by the expressions for the length and width. Let us create a sketch as shown below.



We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$V = l \cdot w \cdot h$$

$$12x^3 + 20x^2 - 21x - 36 = (2x+3)\cdot(3x-4)\cdot h$$

To solve for h, first divide both sides by $2x+3$.

Now, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{6x^3 + 10x^2 - \frac{21}{2}x - 18}{x + \frac{3}{2}}$$

Now, the binomial divisor is $x + \frac{3}{2}$ so $k = -\frac{3}{2}$.

Add each column, multiply the result by $-\frac{3}{2}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 6 & 10 & -\frac{21}{2} & -18 \\ & & -9 & -\frac{3}{2} & 18 \\ \hline & 6 & 1 & -12 & 0 \end{array}$$

Thus, we find $\dfrac{6x^3 + 10x^2 - \frac{21}{2}x - 18}{x + \frac{3}{2}} = 6x^2 + x - 12$

so $\dfrac{(12x^3 + 20x^2 - 21x - 36)}{(2x+3)} = 6x^2 + x - 12$

Now, divide $6x^2 + x - 12$ by $3x - 4$.

We need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 3 to get

$$\frac{2x^2+\frac{1}{3}x-4}{x-\frac{4}{3}}$$

Now, the binomial divisor is $x-\frac{4}{3}$ so $k=\frac{4}{3}$.

Add each column, multiply the result by $\frac{4}{3}$, and repeat until the last column is reached.

$$\begin{array}{c|ccc} \frac{4}{3} & 2 & \frac{1}{3} & -4 \\ & & \frac{8}{3} & 4 \\ \hline & 2 & 3 & 0 \end{array}$$

Thus, we find $\dfrac{2x^2+\frac{1}{3}x-4}{x-\frac{4}{3}}=2x+3$

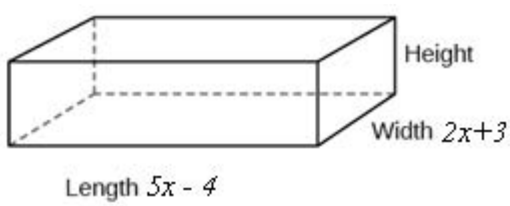
so $\dfrac{6x^2+x-12}{3x-4}=2x+3$

The height of the box is $2x+3$.

68. -

69. Volume is $10x^3+27x^2+2x-24$, length is $5x-4$, width is $2x+3$.

There are a few ways to approach this problem. We need to divide the expression for the volume of the box by the expressions for the length and width. Let us create a sketch as shown below.



We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$V=l\cdot w\cdot h$$

$$10x^3+27x^2+2x-24=(5x-4)\cdot(2x+3)\cdot h$$

To solve for h, first divide both sides by $5x-4$.

Now, we need to change the coefficient of *x* in the divisor 1. Hence, we divide the numerator and denominator by 5 to get

$$\frac{2x^3+\frac{27}{5}x^2+\frac{2}{5}x-\frac{24}{5}}{x-\frac{4}{5}}$$

Now, the binomial divisor is $x-\frac{4}{5}$ so $k=\frac{4}{5}$.

Add each column, multiply the result by $\frac{4}{5}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} \frac{4}{5} & 2 & \frac{27}{5} & \frac{2}{5} & -\frac{24}{5} \\ & & \frac{8}{5} & \frac{28}{5} & \frac{24}{5} \\ \hline & 2 & 7 & 6 & 0 \end{array}$$

Thus, we find $\dfrac{2x^3+\frac{27}{5}x^2+\frac{2}{5}x-\frac{24}{5}}{x-\frac{4}{5}}=2x^2+7x+6$

so $\dfrac{\left(10x^3+27x^2+2x-24\right)}{(5x-4)}=2x^2+7x+6$

Now, divide $2x^2+7x+6$ by $2x+3$.

We need to change the coefficient of *x* in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{x^2+\frac{7}{2}x+3}{x+\frac{3}{2}}$$

Now, the binomial divisor is $x+\frac{3}{2}$ so $k=-\frac{3}{2}$.

Add each column, multiply the result by $-\frac{3}{2}$, and repeat until the last column is reached.

$$\begin{array}{r|rrr} -\frac{3}{2} & 1 & \frac{7}{2} & 3 \\ & & -\frac{3}{2} & -3 \\ \hline & 1 & 2 & 0 \end{array}$$

Thus, we find $\dfrac{x^2 + \frac{7}{2}x + 3}{x + \frac{3}{2}} = x + 2$

so $\dfrac{2x^2 + 7x + 3}{2x + 3} = x + 2$

The height of the box is $x + 2$.

70. -

71. Volume is $\pi(25x^3 - 65x^2 - 29x - 3)$, radius is $5x + 1$.

A cylinder with radius *r* units and height *h* units has a volume of *V* cubic units given by

$V = \pi r^2 h$

Substitute the known values into the formula for the volume of the cylinder.

$\pi(25x^3 - 65x^2 - 29x - 3) = \pi\left(5x + 1\right)^2 h$

To solve for h, first divide both sides by π.

$$\frac{\pi\left(5x+1\right)^2 h}{\pi} = \frac{\pi(25x^3 - 65x^2 - 29x - 3)}{\pi}$$

$$\left(5x+1\right)^2 h = (25x^3 - 65x^2 - 29x - 3)$$

To solve for h, first divide both sides by $5x + 1$.

Now, we need to change the coefficient of *x* in the divisor 1. Hence, we divide the numerator and denominator by 5 to get

$$\frac{5x^3 - 13x^2 - \frac{29}{5}x - \frac{3}{5}}{x + \frac{1}{5}}$$

Now, the binomial divisor is $x + \frac{1}{5}$ so $k = -\frac{1}{5}$.

Add each column, multiply the result by $-\frac{1}{5}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr}
-\frac{1}{5} & 5 & -13 & -\frac{29}{5} & -\frac{3}{5} \\
 & & -1 & \frac{14}{5} & \frac{3}{5} \\
\hline
 & 5 & -14 & -3 & 0
\end{array}$$

Thus, we find $\dfrac{5x^3-13x^2-\frac{29}{5}x-\frac{3}{5}}{x+\frac{1}{5}}=5x^2-14x-3$

so $\dfrac{(25x^3-65x^2-29x-3)}{(5x+1)}=5x^2-14x-3$

Again, divide $5x^2-14x-3$ by $5x+1$.

We need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 5 to get

$$\frac{x^2-\frac{14}{5}x-\frac{3}{5}}{x+\frac{1}{5}}$$

Now, the binomial divisor is $x+\frac{1}{5}$ so $k=-\frac{1}{5}$.

Add each column, multiply the result by $-\frac{1}{5}$, and repeat until the last column is reached.

$$\begin{array}{r|rrr} -\frac{1}{5} & 1 & -\frac{14}{5} & -\frac{3}{5} \\ & & -\frac{1}{5} & \frac{3}{5} \\ \hline & 1 & -3 & 0 \end{array}$$

Thus, we find $\dfrac{x^2-\frac{14}{5}x-\frac{3}{5}}{x+\frac{1}{5}}=x-3$

so $\dfrac{5x^2-14x-3}{5x+1}=x-3$

The height of the cylinder is $x-3$.

72. -

73. Volume is $\pi(3x^4+24x^3+46x^2-16x-32)$, radius is $x+4$.

A cylinder with radius r units and height h units has a volume of V cubic units given by

$$V=\pi r^2 h$$

Substitute the known values into the formula for the volume of the cylinder.

$$\pi(3x^4+24x^3+46x^2-16x-32)=\pi(x+4)^2h$$

To solve for h, first divide both sides by π.

$$\frac{\pi(x+4)^2h}{\pi}=\frac{\pi(3x^4+24x^3+46x^2-16x-32)}{\pi}$$

$$(x+4)^2h=(3x^4+24x^3+46x^2-16x-32)$$

To solve for h, first divide both sides by $x+4$.

Now, the binomial divisor is $x+4$ so $k=-4$.

Add each column, multiply the result by -4, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} -4 & 3 & 24 & 46 & -16 & -32 \\ & & -12 & -48 & 8 & 32 \\ \hline & 3 & 12 & -2 & -8 & 0 \end{array}$$

Thus, $\frac{(3x^4+24x^3+46x^2-16x-32)}{x+4}=3x^3+12x^2-2x-8.$

Again, divide $3x^3+12x^2-2x-8$ by $x+4$.

Now, the binomial divisor is $x+4$ so $k=-4$.

Add each column, multiply the result by -4, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -4 & 3 & 12 & -2 & -8 \\ & & -12 & 0 & 8 \\ \hline & 3 & 0 & -2 & 0 \end{array}$$

The quotient is $3x^2-2$ and the remainder is 0. The height of the cylinder is $3x^2-2$.

Chapter 5
Polynomial and Rational Functions
5.5 Zeros of Polynomial Functions

Verbal

1. Describe a use for the Remainder Theorem.
   The theorem can be used to evaluate a polynomial.
2. -
3. What is the difference between rational and real zeros?
   Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.
4. -
5. If synthetic division reveals a zero, why should we try that value again as a possible solution?
   Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

6. -

7. $\left(3x^3-2x^2+x-4\right)\div\left(x+3\right)$
   To find the functional value using the Remainder Theorem, use synthetic division to divide the polynomial by $x+3$.
   $$\begin{array}{r|rrrr} -3 & 3 & -2 & 1 & -4 \\ & & -9 & 33 & -102 \\ \hline & 3 & -11 & 34 & -106 \end{array}$$
   The remainder is -106. Therefore, $f(-3)=-106$.
8. -
9. $\left(-3x^2+6x+24\right)\div\left(x-4\right)$
   To find the functional value using the Remainder Theorem, use synthetic division to divide the polynomial by $x-4$.
   $$\begin{array}{r|rrr} 4 & -3 & 6 & 24 \\ & & -12 & -24 \\ \hline & -3 & -6 & 0 \end{array}$$
   The remainder is 0. Therefore, $f(4)=0$.

10. -

11. $\left(x^4-1\right)\div\left(x-4\right)$

To find the functional value using the Remainder Theorem, use synthetic division to divide the polynomial by $x-4$.

$$\begin{array}{r|rrrrr} 4 & 1 & 0 & 0 & 0 & -1 \\ & & 4 & 16 & 64 & 256 \\ \hline & 1 & 4 & 16 & 64 & 255 \end{array}$$

The remainder is 255. Therefore, $f(4)=255$.

12. -

13. $\left(4x^3+5x^2-2x+7\right)\div\left(x+2\right)$

To find the functional value using the Remainder Theorem, use synthetic division to divide the polynomial by $x+2$.

$$\begin{array}{r|rrrr} -2 & 4 & 5 & -2 & 7 \\ & & -8 & 6 & -8 \\ \hline & 4 & -3 & 4 & -1 \end{array}$$

The remainder is -1. Therefore, $f(-2)=-1$.

For the following exercises, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

14. -

15. $f(x)=2x^3+x^2-5x+2;\ x+2$

To find the zeros using the Factor Theorem, use synthetic division to divide the polynomial by $x+2$.

$$\begin{array}{r|rrrr} -2 & 2 & 1 & -5 & 2 \\ & & -4 & 6 & -2 \\ \hline & 2 & -3 & 1 & 0 \end{array}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$(x+2)(2x^2-3x+1)$

Factor the quadratic factor to write the polynomial as

$(x+2)(x-1)(2x-1)$

By the Factor Theorem, the zeros of $2x^3+x^2-5x+2$ are –2, 1, and $\frac{1}{2}$.

16. -

17. $f(x) = 2x^3 + 3x^2 + x + 6; \quad x + 2$

To find the zeros using the Factor Theorem, use synthetic division to divide the polynomial by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 2 & 3 & 1 & 6 \\ & & -4 & 2 & -6 \\ \hline & 2 & -1 & 3 & 0 \end{array}$$

The remainder is zero, so $(x + 2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$(x+2)(2x^2 - x + 3)$

For the equation $2x^2 - x + 3 = 0,$ we have $a = 2, b = -1,$ and $c = 3.$ Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} \\ &= \frac{1 \pm \sqrt{1 - 24}}{4} \\ &= \frac{1 \pm i\sqrt{23}}{4} \end{aligned}$$

By the Factor Theorem, the real zero of $2x^3 + 3x^2 + x + 6$ is –2.

18. -

19. $x^3 + 3x^2 + 4x + 12, \; x + 3$

To find the zeros using the Factor Theorem, use synthetic division to divide the polynomial by $x + 3$.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 4 & 12 \\ & & -3 & 0 & -12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

The remainder is zero, so $(x + 3)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$(x - 3)(x^2 + 4)$

Factor the quadratic equation.

$$\begin{aligned} x^2+4&=0\\ x^2&=-4\\ x&=\pm 2i \end{aligned}$$

By the Factor Theorem, the real zero of $x^3+3x^2+4x+12$ is -3.

20. -

21. $2x^3+5x^2-12x-30,\ 2x+5$

First, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$$\frac{x^3+\frac{5}{2}x^2-6x-15}{x+\frac{5}{2}}$$

Now, the binomial divisor is $x+\frac{5}{2}$ so $k=-\frac{5}{2}$.

Add each column, multiply the result by $-\frac{5}{2}$, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -\frac{5}{2} & 1 & \frac{5}{2} & -6 & -15 \\ & & -\frac{5}{2} & 0 & 15 \\ \hline & 1 & 0 & -6 & 0 \end{array}$$

Thus, we find $\dfrac{x^3+\frac{5}{2}x^2-6x-15}{x+\frac{5}{2}}=x^2-6$

so $\dfrac{2x^3+5x^2-12x-30}{2x+5}=x^2-6$

The remainder is zero, so $(2x+5)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$(2x+5)(x^2-6)$

Factor the quadratic equation.

$$\begin{aligned} x^2-6&=0\\ x^2&=6\\ x&=\pm\sqrt{6} \end{aligned}$$

By the Factor Theorem, the zeros of $2x^3+5x^2-12x-30$ are $-\frac{5}{2},\ \sqrt{6},\ -\sqrt{6}$.

For the following exercises, use the Rational Zero Theorem to find all real zeros.

22. -
23. $2x^3 + 7x^2 - 10x - 24 = 0$

    The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –24 and q is a factor of 2.

    $$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of } -24}{\text{factor of 2}}$$

    The factors of 2 are ± 1 and ± 2 and the factors of 24 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12$ and $\pm 24.$ The possible values for $\frac{p}{q}$ are $\pm\frac{1}{2}, \pm 1, \pm 2, \pm\frac{3}{2}, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12$ and $\pm 24.$

    These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 2.

    $$\begin{array}{r|rrrr} 2 & 2 & 7 & -10 & -24 \\ & & 4 & 22 & 24 \\ \hline & 2 & 11 & 12 & 0 \end{array}$$

    Dividing by $(x-2)$ gives a remainder of 0, so 2 is a zero of the function. The polynomial can be written as $(x-2)(2x^2 + 11x + 12).$

    We can factor the quadratic factor to write the polynomial as

    $(2x^2 + 11x + 12) = (2x+3)(x+4)$

    To find the other zeros, set each factor equal to 0.

    $$\begin{array}{lcl} (2x+3) = 0 & \text{or} & (x+4) = 0 \\ x = -\frac{3}{2} & & x = -4 \end{array}$$

    The zeros of $2x^3 + 7x^2 - 10x - 24 = 0$ are $2, -4, -\frac{3}{2}.$ Note that these were all listed as candidates from the Rational Zero Theorem

24. -
25. $x^3 + 5x^2 - 16x - 80 = 0$

    The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –80 and q is a factor of 1.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of } -80}{\text{factor of 1}}$$

The factors of 1 are ± 1 and the factors of –18 are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40$ and $\pm 80.$ The possible values for $\frac{p}{q}$ are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 16, \pm 20, \pm 40$ and $\pm 80.$

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 4.

$$\begin{array}{r|rrrr} 4 & 1 & 5 & -16 & -80 \\ & & 4 & 36 & 80 \\ \hline & 1 & 9 & 20 & 0 \end{array}$$

Dividing by $(x-4)$ gives a remainder of 0, so 4 is a zero of the function. The polynomial can be written as

$(x-4)(x^2+9x+20).$

We can factor the quadratic factor to write the polynomial as

$(x^2+9x+20)=(x+4)(x+5)$

To find the other zeros, set each factor equal to 0.

$$\begin{array}{lcl} (x+4)=0 & \text{or} & (x+5)=0 \\ x=-4 & & x=-5 \end{array}$$

The zeros of $x^3+5x^2-16x-80=0$ are $4, -4, -5.$ Note that these were all listed as candidates from the Rational Zero Theorem.

26. -

27. $2x^3-3x^2-32x-15=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x),$ then p is a factor of –15 and q is a factor of 2.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of } -15}{\text{factor of 2}}$$

The factors of 2 are ± 1 and ± 2 and the factors of –15 are $\pm 1, \pm 3, \pm 5$ and $\pm 15.$ The possible values for $\frac{p}{q}$ are $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm\frac{5}{2}, \pm 5, \pm\frac{15}{2}$ and $\pm 15.$

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with $-\frac{1}{2}$.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -3 & -32 & -15 \\ & & -1 & 2 & 15 \\ \hline & 2 & -4 & -30 & 0 \end{array}$$

Dividing by $\left(x+\frac{1}{2}\right)$ gives a remainder of 0, so $-\frac{1}{2}$ is a zero of the function. The polynomial can be written as

$$\begin{aligned} \left(x+\frac{1}{2}\right)(2x^2-4x-30) &= 2\left(x+\frac{1}{2}\right)(x^2-2x-15) \\ &= (2x+1)(x^2-2x-15) \end{aligned}$$

We can factor the quadratic factor to write the polynomial as

$(x^2-2x-15)=(x-5)(x+3)$

To find the other zeros, set each factor equal to 0.

$$\begin{array}{lll} (x+3)=0 & \text{or} & (x-5)=0 \\ x=-3 & & x=5 \end{array}$$

The zeros of $2x^3-3x^2-32x-15=0$ are $-3, 5, -\frac{1}{2}$. Note that these were all listed as candidates from the Rational Zero Theorem.

28. -

29. $2x^3-3x^2-x+1=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of 1}}{\text{factor of 2}}$$

The factors of 2 are ± 1 and ± 2 and the factors of 1 are ± 1. The possible values for $\frac{p}{q}$ are $\pm\frac{1}{2}$ and ± 1.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with $\frac{1}{2}$.

$$\begin{array}{c|cccc} \frac{1}{2} & 2 & -3 & -1 & 1 \\ & & 1 & -1 & -1 \\ \hline & 2 & -2 & -2 & 0 \end{array}$$

Dividing by $\left(x-\frac{1}{2}\right)$ gives a remainder of 0, so $\frac{1}{2}$ is a zero of the function. The polynomial can be written as

$$\begin{aligned}\left(x-\frac{1}{2}\right)(2x^2-2x-2) &= 2\left(x-\frac{1}{2}\right)(x^2-x-1) \\ &= (2x-1)(x^2-x-1)\end{aligned}$$

For the equation $x^2-x-1=0$, we have $a=1, b=-1, \text{and } c=-1.$ Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{1\pm\sqrt{(-1)^2-4\cdot1\cdot(-1)}}{2\cdot1} \\ &= \frac{1\pm\sqrt{1+4}}{2} \\ &= \frac{1\pm\sqrt{5}}{2}\end{aligned}$$

The zeros of $2x^3-3x^2-x+1=0$ are $\frac{1}{2},\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}$. Note that $\frac{1}{2}$ was all listed as a candidate from the Rational Zero Theorem.

30. -

31. $2x^3-5x^2+9x-9=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –9 and q is a factor of 2.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of } -9}{\text{factor of } 2}$$

The factors of 2 are ± 1 and ± 2 and the factors of –9 are $\pm 1, \pm 3$ and ± 9. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm 3, \pm\frac{9}{2}$ and ± 9.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with $\frac{3}{2}$.

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -5 & 9 & -9 \\ & & 3 & -3 & 9 \\ \hline & 2 & -2 & 6 & 0 \end{array}$$

Dividing by $\left(x-\frac{3}{2}\right)$ gives a remainder of 0, so $\frac{3}{2}$ is a zero of the function. The polynomial can be written as

$$\begin{aligned}\left(x-\frac{3}{2}\right)(2x^2-2x+6) &= 2\left(x-\frac{3}{2}\right)(x^2-x+3)\\ &= (2x-3)(x^2-x+3)\end{aligned}$$

For the equation $x^2-x+3=0$, we have $a=1, b=-1,$ and $c=3$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &= \frac{1\pm\sqrt{(-1)^2-4\cdot 1\cdot(3)}}{2\cdot 1}\\ &= \frac{1\pm\sqrt{1-12}}{2}\\ &= \frac{1\pm i\sqrt{12}}{2}\end{aligned}$$

Thus, the real zero of $2x^3-5x^2+9x-9=0$ is $\frac{3}{2}$. Note that $\frac{3}{2}$ was all listed as a candidate from the Rational Zero Theorem.

32. -

33. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

The Rational Zero Theorem tells us that if $\dfrac{p}{q}$ is a zero of $f(x)$, then p is a factor of 12 and q is a factor of 1.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of 12}}{\text{factor of 1}}$$

The factors of 1 are ±1 and the factors of 12 are $\pm1, \pm2, \pm3, \pm4, \pm6$ and ±12. The possible values for $\dfrac{p}{q}$ are $\pm1, \pm2, \pm3, \pm4, \pm6$ and ±12.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with -1.

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & -7 & 8 & 12 \\ & & -1 & 3 & 4 & -12 \\ \hline & 1 & -3 & -4 & 12 & 0 \end{array}$$

Dividing by $(x+1)$ gives a remainder of 0, so -1 is a zero of the function. The polynomial can be written as $(x+1)(x^3 - 3x^2 - 4x + 12)$.

Again, use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's consider 2.

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -4 & 12 \\ & & 2 & -2 & -12 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Dividing by $(x-2)$ gives a remainder of 0, so 2 is a zero of the function. The polynomial can be written as

$(x+1)(x-2)(x^2 - x - 6).$

We can factor the quadratic factor to write the polynomial as

$(x^2 - x - 6) = (x-3)(x+2)$

To find the other zeros, set each factor equal to 0.

$(x-3) = 0 \quad \text{or} \quad (x+2) = 0$

$x = 3 \qquad\qquad x = -2$

The zeros of $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$ are $2, 3, -1, -2$. Note that these were all listed as candidates from the Rational Zero Theorem.

34. -

35. $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 6 and q is a factor of 4.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of 6}}{\text{factor of 4}}$$

The factors of 4 are $\pm 1, \pm 2$ and ± 4 and the factors of 6 are $\pm 1, \pm 2, \pm 3$ and ± 6. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}$ and ± 6.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 2.

$$\begin{array}{r|rrrrr} 2 & 4 & 4 & -25 & -1 & 6 \\ & & 8 & 24 & -2 & -6 \\ \hline & 4 & 12 & -1 & -3 & 0 \end{array}$$

Dividing by $(x-2)$ gives a remainder of 0, so 2 is a zero of the function. The polynomial can be written as

$(x-2)(4x^3 + 12x^2 - x - 3)$

Again, use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's consider -3.

$$\begin{array}{r|rrrr} -3 & 4 & 12 & -1 & -3 \\ & & -12 & 0 & 3 \\ \hline & 4 & 0 & -1 & 0 \end{array}$$

Dividing by $(x+3)$ gives a remainder of 0, so -3 is a zero of the function. The polynomial can be written as $(x-2)(x+3)(4x^2-1)$.

The quadratic is a different of squares. We can factor it to write the polynomial as $(4x^2-1) = (2x+1)(2x-1)$.

To find the other zeros, set each factor equal to 0.

$$2x+1=0 \quad \text{or} \quad 2x-1=0$$

$$x = -\frac{1}{2} \qquad\qquad x = \frac{1}{2}$$

The zeros of $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$ are $\frac{1}{2}, -\frac{1}{2}, 2, -3.$ Note that these were all listed as candidates from the Rational Zero Theorem.

36. -

37. $x^4+2x^3-4x^2-10x-5=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –5 and q is a factor of 1.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of }-5}{\text{factor of 1}}$$

The factors of 1 are ±1 and the factors of –5 are ±1 and ±5. The possible values for $\frac{p}{q}$ are ±1 and ±5.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with -1.

$$\begin{array}{r|rrrrr} -1 & 1 & 2 & -4 & -10 & -5 \\ & & -1 & -1 & 5 & 5 \\ \hline & 1 & 1 & -5 & -5 & 0 \end{array}$$

Dividing by $(x+1)$ gives a remainder of 0, so -1 is a zero of the function. The polynomial can be written as $(x+1)(x^3+x^2-5x-5)$.

Again, use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's consider -1.

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -5 & -5 \\ & & -1 & 0 & 5 \\ \hline & 1 & 0 & -5 & 0 \end{array}$$

Dividing by $(x+1)$ gives a remainder of 0, so -1 is a zero of the function. The polynomial can be written as $(x+1)(x+1)(x^2-5)$.

To find the other zeros, set the factor equal to 0.

$$\begin{aligned} x^2-5&=0\\ x^2&=5\\ x&=\pm\sqrt{5} \end{aligned}$$

The zeros of $x^4+2x^3-4x^2-10x-5=0$ are $-1,-1,\sqrt{5},-\sqrt{5}$. Note that 1 and -1were listed as candidates from the Rational Zero Theorem.

38. -

39. $8x^4+26x^3+39x^2+26x+6=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 6 and q is a factor of 8.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of 6}}{\text{factor of 8}}$$

The factors of 6 are $\pm1, \pm2, \pm3$ and ± 6 and the factors of 8 are $\pm1, \pm2, \pm4$ and ± 8. The possible values for $\frac{p}{q}$ are $\pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm2, \pm\frac{2}{3}, \pm4, \pm\frac{4}{3}, \pm8$ and $\pm\frac{8}{3}$.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with $-\frac{1}{2}$.

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 8 & 26 & 39 & 26 & 6 \\ & & -4 & -11 & -14 & -6 \\ \hline & 8 & 22 & 28 & 12 & 0 \end{array}$$

Dividing by $\left(x+\frac{1}{2}\right)$ gives a remainder of 0, so $-\frac{1}{2}$ is a zero of the function. The polynomial can be written as

$$\begin{aligned} \left(x+\frac{1}{2}\right)(8x^3+22x^2+28x+12) &= 2\left(x+\frac{1}{2}\right)(4x^3+11x^2+14x+6) \\ &= \left(2x+1\right)(4x^3+11x^2+14x+6) \end{aligned}$$

Again, use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's consider $-\frac{3}{4}$.

$$\begin{array}{r|rrrr} -\frac{3}{4} & 4 & 11 & 14 & 6 \\ & & -3 & -6 & -6 \\ \hline & 4 & 8 & 8 & 0 \end{array}$$

Dividing by $\left(x+\frac{3}{4}\right)$ gives a remainder of 0, so $-\frac{3}{4}$ is a zero of the function. The polynomial can be written as

$$\begin{aligned} (2x+1)\left(x+\frac{3}{4}\right)(4x^2+8x+8) &= 4(2x+1)\left(x+\frac{3}{4}\right)(x^2+2x+2) \\ &= (2x+1)\left(4x+3\right)(x^2+2x+2) \end{aligned}$$

For the equation $x^2+2x+2=0,$ we have $a=1, b=2, \text{and } c=2.$ Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot 2}}{2\cdot 1} \\ &= \frac{-2 \pm \sqrt{4-8}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i \end{aligned}$$

Thus, the real zeros of $8x^4+26x^3+39x^2+26x+6=0$ are $-\frac{3}{4}, -\frac{1}{2}.$ Note that these were listed as candidates from the Rational Zero Theorem.

For the following exercises, find all complex solutions (real and non-real).

40. -
41. $x^3-8x^2+25x-26=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –26 and q is a factor of 1.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of } -26}{\text{factor of } 1}$$

The factors of 1 are ± 1 and the factors of –26 are $\pm 1, \pm 2, \pm 13 \text{ and } \pm 26.$ The possible values for $\frac{p}{q}$ are $\pm 1, \pm 2, \pm 13 \text{ and } \pm 26.$

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 2.

$$\begin{array}{r|rrrr} 2 & 1 & -8 & 25 & -26 \\ & & 2 & -12 & 26 \\ \hline & 1 & -6 & 13 & 0 \end{array}$$

Dividing by $(x-2)$ gives a remainder of 0, so 2 is a zero of the function. The polynomial can be written as $(x-2)(x^2-6x+13)$.

For the equation $x^2-6x+13=0,$ we have $a=1, b=-6, \text{and } c=13.$ Substituting these values into the formula we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1}$$
$$= \frac{6 \pm \sqrt{36 - 52}}{2}$$
$$= \frac{6 \pm 4i}{2}$$
$$= 3 \pm 2i$$

Thus, the zeros of $x^3 - 8x^2 + 25x - 26 = 0$ are $2, 3+2i, 3-2i.$

42. -

43. $3x^3 - 4x^2 + 11x + 10 = 0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 10 and q is a factor of 3.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of 10}}{\text{factor of 3}}$$

The factors of 3 are ± 1 and ± 3 and the factors of 10 are $\pm 1, \pm 2, \pm 5$ and ± 10. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 2, \pm\frac{5}{3}, \pm 5, \pm\frac{10}{3}$ and ± 10.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with $-\frac{2}{3}$.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & -4 & 11 & 10 \\ & & -2 & 4 & -10 \\ \hline & 3 & -6 & 15 & 0 \end{array}$$

Dividing by $\left(x + \frac{2}{3}\right)$ gives a remainder of 0, so $-\frac{2}{3}$ is a zero of the function. The polynomial can be written as

$$\left(x + \frac{2}{3}\right)(3x^2 - 6x + 15) = 3\left(x + \frac{2}{3}\right)(x^2 - 2x + 5)$$
$$= (3x + 2)(x^2 - 2x + 5)$$

For the equation $x^2 - 2x + 5 = 0,$ we have $a = 1, b = -2, \text{and } c = 5.$ Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \end{aligned}$$

Thus, the zeros of $3x^3 - 4x^2 + 11x + 10 = 0$ is $-\frac{2}{3}, 1+2i, 1-2i.$

44. -

45. $2x^3 - 3x^2 + 32x + 17 = 0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x),$ then p is a factor of 17 and q is a factor of 2.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of 17}}{\text{factor of 2}}$$

The factors of 2 are ± 1 and ± 2 and the factors of 10 are ± 1 and $\pm 17.$ The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{17}{2}$ and $\pm 17.$

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let’s begin with $-\frac{1}{2}.$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -3 & 32 & 17 \\ & & -1 & 2 & -17 \\ \hline & 2 & -4 & 34 & 0 \end{array}$$

Dividing by $\left(x + \frac{1}{2}\right)$ gives a remainder of 0, so $-\frac{1}{2}$ is a zero of the function. The polynomial can be written as

$$\left(x+\frac{1}{2}\right)(2x^2-4x+34)=2\left(x+\frac{1}{2}\right)(x^2-2x+17)$$
$$=\left(2x+1\right)(x^2-2x+17)$$

For the equation $x^2-2x+17=0$, we have $a=1, b=-2,$ and $c=17.$ Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{2\pm\sqrt{(-2)^2-4\cdot 1\cdot 17}}{2\cdot 1} \\ &= \frac{2\pm\sqrt{4-68}}{2} \\ &= \frac{2\pm 8i}{2} \\ &= 1\pm 4i \end{aligned}$$

Thus, the zeros of $2x^3-3x^2+32x+17=0$ is $-\frac{1}{2}, 1+4i, 1-4i.$ Note that $-\frac{1}{2}.$ was listed as a candidate from the Rational Zero Theorem.

Graphical

For the following exercises, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.

46. -

47. $f(x)=x^4-x^2-1$

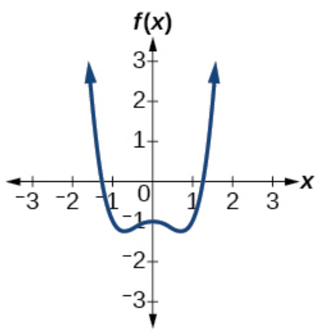
Begin by determining the number of sign changes.

There is only one sign change, so there is 1 positive real root.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= (-x)^4-(-x)^2-1 \\ &= x^4-x^2-1 \end{aligned}$$

There is only one sign change, so there is 1 negative real root.



The graph shows that there are 1 positive real zero and 1 negative real zero.

48. -

49. $f(x) = x^3 - 2x^2 + x - 1$

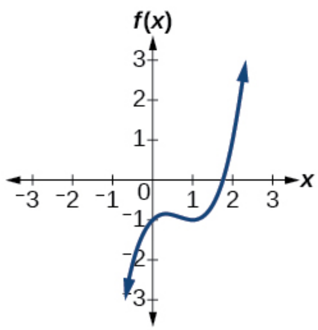
Begin by determining the number of sign changes.

There are three sign changes, so there are either 3 or 1 positive real roots.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= (-x)^3 - 2(-x)^2 + (-x) - 1 \\ &= -x^3 - 2x^2 - x - 1 \end{aligned}$$

In this case, $f(-x)$ has no sign change. This tells us that $f(x)$ has 0 negative real zeros.



The graph shows that there are 1 positive real zero and 0 negative real zeros.

50. -

51. $f(x) = 2x^3 + 37x^2 + 200x + 300$

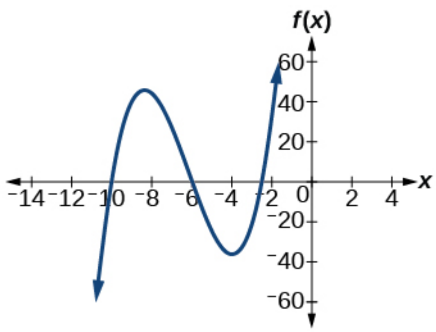
Begin by determining the number of sign changes.

In this case, $f(x)$ has no sign change. This tells us that $f(x)$ has 0 positive real zeros.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= 2(-x)^3 + 37(-x)^2 + 200(-x) + 300 \\ &= -2x^3 + 37x^2 - 200x + 300 \end{aligned}$$

There are three sign changes, so there are either 3 or 1 negative real roots.



The graph shows that there are 0 positive real zeros and 3 negative real zeros.

52. -

53. $f(x)=2x^4-5x^3-5x^2+5x+3$

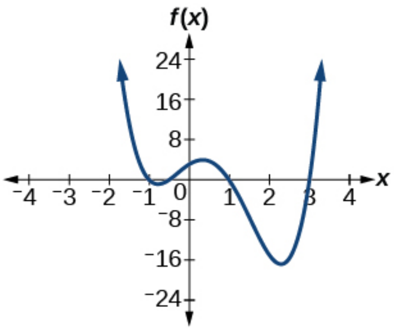
Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots.

Next, we examining t $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= 2(-x)^4-5(-x)^3-5(-x)^2+5(-x)+3 \\ &= 2x^4+5x^3-5x^2-5x+3 \end{aligned}$$

There are two sign changes, so there are either 2 or 0 negative real roots.



The graph shows that there are 2 positive real zeros and 2 negative real zeros.

54. -

55. $f(x)=10x^4-21x^2+11$

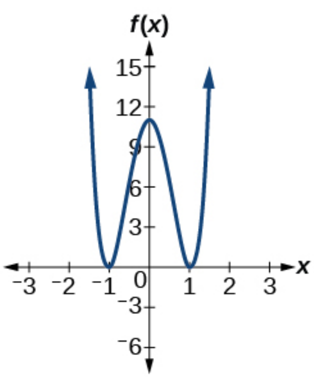
Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= 10(-x)^4-21(-x)^2+11 \\ &= 10x^4-21x^2+11 \end{aligned}$$

There are two sign changes, so there are either 2 or 0 negative real roots.



The graph shows that there are 2 positive real zeros and 2 negative real zeros.

Numeric

For the following exercises, list all possible rational zeros for the functions.

56. -

57. $f(x)=2x^3+3x^2-8x+5$

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, 5, and the factors of the leading coefficient, 2.

The constant term is 5; the factors of 5 are $p=\pm 1, \pm 5$.

The leading coefficient is 2; the factors of 2 are $q=\pm 1, \pm 2$.

If any of the three real zeros are rational zeros, then they will be of one of the following factors of 5 divided by one of the factors of 2.

$\frac{p}{q}=\pm\frac{1}{1}, \pm\frac{5}{1} \quad \frac{p}{q}=\pm\frac{1}{2}, \pm\frac{5}{2}$

Thus, the possible values for $\frac{p}{q}$ are $\pm\frac{1}{2}, \pm 1, \pm 5, \pm\frac{5}{2}$.

58. -

59. $f(x)=6x^4-10x^2+13x+1$

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, 1, and the factors of the leading coefficient, 6.

The constant term is 1; the factors of 1 are $p=\pm 1$.

The leading coefficient is 6; the factors of 6 are $q=\pm 1, \pm 2, \pm 3, \pm 6$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of 1 divided by one of the factors of 6.

Thus, the possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$.

60. -

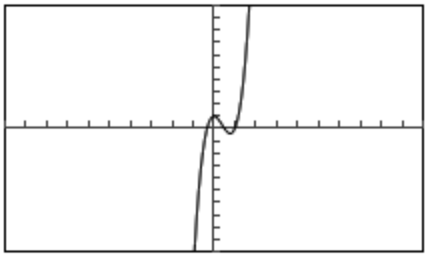
Technology

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

61. $f(x)=6x^3-7x^2+1$

Press $\boxed{Y=}$ $\boxed{6}\boxed{x}\boxed{\wedge}\boxed{3}$ $\boxed{-}$ $\boxed{7}\boxed{x}$ $\boxed{\wedge}$ $\boxed{2}$ $\boxed{+}$ $\boxed{1}$

This graphs the curve in a standard window, similar to the graph below.



Remember that the real zeroes of a polynomial correspond to the *x*-intercepts of the graph of that polynomial.

To find *x*-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the *x*-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the *x*-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the *x*-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The *x*-value of the *x*-intercept is given.

Repeat this process for the other *x*-intercept.

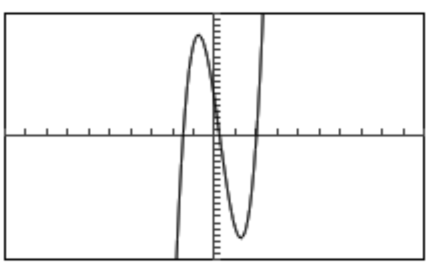
The *x*-intercepts are $1, \frac{1}{2}, -\frac{1}{3}$.

62. -

63. $f(x) = 8x^3 - 6x^2 - 23x + 6$

Press $\boxed{Y=}$ $\boxed{8}\boxed{x}\boxed{\wedge}\boxed{3}$ $\boxed{-}$ $\boxed{6}\boxed{x}\boxed{\wedge}\boxed{2}$ $\boxed{-}$ $\boxed{23}\boxed{x}$ $\boxed{+}$ $\boxed{6}$

This graphs the curve in a standard window, similar to the graph below.



Remember that the real zeroes of a polynomial correspond to the *x*-intercepts of the graph of that polynomial.

To find *x*-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercept.

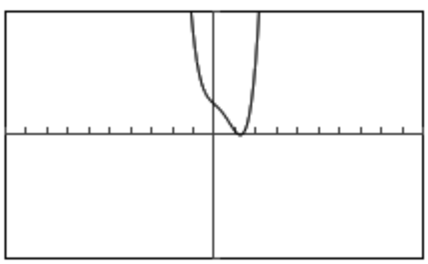
The x-intercepts are $2, \frac{1}{4}, -\frac{3}{2}$.

64. -

65. $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

Press $\boxed{Y=}$ $\boxed{16}\boxed{x}\boxed{\wedge}\boxed{4}$ $\boxed{-}$ $\boxed{24}\boxed{x}\boxed{\wedge}\boxed{3}$ $\boxed{+}$ $\boxed{x}\boxed{\wedge}\boxed{2}$ $\boxed{-}$ $\boxed{15}\boxed{x}$ $\boxed{+}$ $\boxed{25}$

This graphs the curve in a standard window, similar to the graph below.



Remember that the real zeroes of a polynomial correspond to the x-intercepts of the graph of that polynomial.

To find x-intercepts, press $\boxed{2^{nd}}$ CALC (for calculate, above $\boxed{\text{TRACE}}$ $\boxed{2}$ (for zero.)

The calculator asks for a left bound. Move the cursor to the left so that it is just to the left of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

The calculator asks for a right bound. Move the cursor to the right so that it is just to the right of the x-intercept that we want. Press $\boxed{\text{ENTER}}$.

When you are asked for your guess, move as close as you can to the x-intercept and press $\boxed{\text{ENTER}}$. Notice that the word "zero" is now on the screen. The x-value of the x-intercept is given.

Repeat this process for the other x-intercept.

The x-intercept is $\frac{5}{4}$.

Extensions

For the following exercises, construct a polynomial function of least degree possible using the given information.

66. -

67. Real roots: –1 with multiplicity 2 and 1 and $\left(2, f\left(2\right)\right) = \left(2, 4\right)$

The polynomial must have factors of $(x+1)^2$ and $(x-1)$.

Let's begin by multiplying these factors.

$$\begin{aligned} f(x) &= a(x+1)^2(x-1) \\ &= a(x^2+2x+1)(x-1) \\ &= a(x^3+x^2-x-1) \end{aligned}$$

We need to find a to ensure $f(2)=4.$ Substitute $x=2$ and $f(2)=4$ into $f(x)$.

$$4 = a(2^3+2^2-2-1)$$
$$4 = a(8+4-2-1)$$
$$4 = 9a$$
$$a = \frac{4}{9}$$

So the polynomial function is $f(x) = \dfrac{4}{9}(x^3+x^2-x-1)$.

68. -

69. Real roots: $-\dfrac{1}{2}$, 0, $\dfrac{1}{2}$ and $\left(-2, f\left(-2\right)\right) = \left(-2, 6\right)$

The polynomial must have factors of $x, \left(x+\dfrac{1}{2}\right)$ and $\left(x-\dfrac{1}{2}\right)$.

Let's begin by multiplying these factors.

$$\begin{aligned} f(x) &= ax\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right) \\ &= \frac{a}{4}x\left(2x-1\right)\left(2x+1\right) \\ &= \frac{a}{4}x\left(4x^2-1\right) \\ &= \frac{a}{4}\left(4x^3-x\right) \end{aligned}$$

We need to find a to ensure $f(-2)=6.$ Substitute $x=-2$ and $f(-2)=6$ into $f(x)$.

$$6 = \frac{a}{4}\left(4(-2)^3 - (-2)\right)$$
$$24 = a(-32+2)$$
$$24 = -30a$$
$$a = -\frac{4}{5}$$

So the polynomial function is

$$f(x) = -\frac{4}{5} \times \frac{1}{4}\left(4x^3 - x\right)$$
$$= -\frac{1}{5}\left(4x^3 - x\right)$$

70. -

Real-World Applications

For the following exercises, find the dimensions of the box described.

71. The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

Begin by writing an equation for the volume of the box. The volume of a rectangular solid is given by $V = lwh.$ We were given that the length must be twice as long as the width, so we can express the length of the box as $l = 2w.$ We were given that the height of the box is 2 inches greater than the width, so we can express the height of the box as $h = w + 2.$ Let's write the volume of the box in terms of width of the box.

$$V = 2w(w)(w+2)$$
$$V = 2w^3 + 4w^2$$

Substitute the given volume into this equation.

$192 = 2w^3 + 4w^2$ Substitute 192 for V.

$96 = w^3 + 2w^2$ Divide both sides by 2.

$0 = w^3 + 2w^2 - 96$ Subtract 96 from both sides.

The Rational Zero Theorem tells us that the possible rational zeros are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24, \pm32, \pm48$ and $\pm96.$

Only positive numbers make sense as dimensions for a box, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small box. Use synthetic division to check $w = 1.$

$$\begin{array}{r|rrrr} 1 & 1 & 2 & 0 & -96 \\ & & 1 & 3 & 3 \\ \hline & 1 & 3 & 3 & -93 \end{array}$$

Since 1 is not a solution, we will check $w = 2.$

$$\begin{array}{r|rrrr} 2 & 1 & 2 & 0 & -96 \\ & & 2 & 8 & 16 \\ \hline & 1 & 4 & 8 & -80 \end{array}$$

Since 2 is not a solution either, we will test $w = 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 2 & 0 & -96 \\ & & 3 & 15 & 45 \\ \hline & 1 & 5 & 15 & -41 \end{array}$$

Since 3 is not a solution either, we will test $w = 4$.

$$\begin{array}{r|rrrr} 4 & 1 & 2 & 0 & -96 \\ & & 4 & 24 & 96 \\ \hline & 1 & 6 & 24 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 4 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the box.

$l = 2w = 2 \cdot 4 = 8$ and $h = w + 2 = 4 + 2 = 6$

The box should have dimensions 8 inches by 4 inches by 6 inches.

72. -

73. The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

Begin by writing an equation for the volume of the box. The volume of a rectangular solid is given by $V = lwh.$ We were given that the length is one inch more than the width, so we can express the length of the box as $l = w + 1.$ We were given that the width of the box is one inch greater than the height, so we can express the height of the box as $w = h + 1.$ Let's write the volume of the box in terms of height of the box.

$$\begin{aligned} V &= (h+2)(h+1)h \\ &= \left(h^2 + 3h + 2\right)h \\ &= h^3 + 3h^2 + 2h \end{aligned}$$

Substitute the given volume into this equation.

$86.625 = h^3 + 3h^2 + 2h$ Substitute 86.625 for V.

$0 = h^3 + 3h^2 + 2h - 86.625$ Subtract 86.625 from both sides.

Let's begin by testing values that make the most sense as dimensions for a small box. Use synthetic division to check $h = 1.$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 2 & -86.625 \\ & & 1 & 4 & 6 \\ \hline & 1 & 4 & 6 & -80.625 \end{array}$$

Since 1 is not a solution, we will use the factor greater than 1. Consider $h = 1.5.$

$$\begin{array}{r|rrrr} 1.5 & 1 & 3 & 2 & -86.625 \\ & & 1.5 & 6.75 & 13.125 \\ \hline & 1 & 4.5 & 8.75 & -73.5 \end{array}$$

Since 1.5 is not a solution either, we will test $h = 3.5.$

$$\begin{array}{r|rrrr} 3.5 & 1 & 3 & 2 & -86.625 \\ & & 3.5 & 22.75 & 86.625 \\ \hline & 1 & 6.5 & 24.75 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 3.5 is a solution to the equation. We can use the relationships between the height and the other dimensions to determine the length and width of the box.

$w = h + 1 = 3.5 + 1 = 4.5$ and $l = w + 1 = 4.5 + 1 = 5.5$

The box should have dimensions 5.5 inches by 4.5 inches by 3.5 inches.

74. -

75. The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

Begin by writing an equation for the volume of the box. The volume of a rectangular solid is given by $V = lwh.$ We were given that the length must be three inches more than the width, so we can express the length of the box as $l = w + 3.$ We were given that the width of the box is 2 inches more than the height, so we can express the width of the box as $w = h + 2.$ Let's write the volume of the box in terms of height of the box.

$$\begin{aligned} V &= (h+5)(h+2)h \\ &= \left(h^2 + 7h + 10\right)h \\ &= h^3 + 7h^2 + 10h \end{aligned}$$

Substitute the given volume into this equation.

$120 = h^3 + 7h^2 + 10h$ Substitute 120 for V.

$0 = h^3 + 7h^2 + 10h - 120$ Subtract 120 from both sides.

The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 4,$ $\pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 30, \ \pm 40, \ \pm 60$ and $\pm 120.$

Only positive numbers make sense as dimensions for a box, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small box. Use synthetic division to check $h = 1.$

$$\begin{array}{r|rrrr} 1 & 1 & 7 & 10 & -120 \\ & & 1 & 8 & 18 \\ \hline & 1 & 8 & 18 & -102 \end{array}$$

Since 1 is not a solution, we will check $h = 2$.

$$\begin{array}{r|rrrr} 2 & 1 & 7 & 10 & -120 \\ & & 2 & 18 & 56 \\ \hline & 1 & 9 & 28 & -64 \end{array}$$

Since 2 is not a solution either, we will test $h = 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 7 & 10 & -120 \\ & & 3 & 30 & 120 \\ \hline & 1 & 10 & 40 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 3 is a solution to the equation. We can use the relationships between the height and the other dimensions to determine the length and width of the box.

$w = h + 2 = 3 + 2 = 5$ and $l = w + 3 = 5 + 3 = 8$

The box should have dimensions 8 inches by 5 inches by 3 inches.

For the following exercises, find the dimensions of the right circular cylinder described.

76. -

77. The height is one less than one half the radius. The volume is 72π cubic meters.

A cylinder with radius r units and height h units has a volume of V cubic units given by $V = \pi r^2 h$. We were given that the height is one less than one half the radius, so we can express the height of the right circular cylinder as $h = \frac{r}{2} - 1$.

Rewrite the expression of the radius in terms of the height.

$$\frac{r}{2} = h + 1$$

$$r = 2(h + 1)$$

Let's write the volume of the right circular cylinder in terms of the height.

$$\begin{aligned} V &= \pi\left[2(h+1)\right]^2 h \\ &= 4\pi\left(h^2 + 2h + 1\right)h \\ &= 4\pi\left(h^3 + 2h^2 + h\right) \end{aligned}$$

Substitute the given volume into this equation.

$$72\pi = 4\pi\left(h^3 + 2h^2 + h\right) \qquad \text{Substitute } 72\pi \text{ for } V.$$
$$18 = h^3 + 2h^2 + h \qquad \text{Divide } 4\pi \text{ by both sides.}$$
$$0 = h^3 + 2h^2 + h - 18 \qquad \text{Subtract 18 from both sides.}$$

The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$ and ± 18.

Only positive numbers make sense as dimensions for a right circular cylinder, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a right circular cylinder. Use synthetic division to check $h = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 2 & 1 & -18 \\ & & 1 & 3 & 4 \\ \hline & 1 & 3 & 4 & -14 \end{array}$$

Since 1 is not a solution, we will check $h = 2$.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & 1 & -18 \\ & & 2 & 8 & 18 \\ \hline & 1 & 4 & 9 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 2 is a solution to the equation. We can use the relationships between the height and the radius to determine the radius of the right circular cylinder.

$$r = 2(h+1) = 2(2+1) = 6$$

The right circular cylinder should have a radius 6 meters and a height 2 meters.

78. -

79. The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.

A cylinder with radius *r* units and height *h* units has a volume of *V* cubic units given by $V = \pi r^2 h$. We were given that the radius and height differ by two meters and height is larger, so we can express the height of the right circular cylinder as $h = r + 2$. Let's write the volume of the right circular cylinder in terms of the radius.

$$V = \pi r^2 (r+2)$$
$$= \pi\left(r^3 + 2r^2\right)$$

Substitute the given volume into this equation.

$$28.125\pi = \pi\left(r^3 + 2r^2\right) \qquad \text{Substitute } 28.125\pi \text{ for } V.$$
$$28.125 = r^3 + 2r^2 \qquad \text{Divide } \pi \text{ by both sides.}$$
$$0 = r^3 + 2r^2 - 28.125 \qquad \text{Subtract 28.125 from both sides.}$$

Let's begin by testing values that make the most sense as dimensions for a right circular cylinder. Use synthetic division to check $r = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 2 & 0 & -28.125 \\ & & 1 & 3 & 3 \\ \hline & 1 & 3 & 3 & -25.125 \end{array}$$

Since 1 is not a solution, we will check the next factor, say $r = 1.5$.

$$\begin{array}{r|rrrr} 1.5 & 1 & 2 & 0 & -28.125 \\ & & 1.5 & 5.25 & 7.875 \\ \hline & 1 & 3.5 & 5.25 & -20.25 \end{array}$$

Since 1.5 is not a solution, we will check the next factor, say $r = 2.5$.

$$\begin{array}{r|rrrr} 2.5 & 1 & 2 & 0 & -28.125 \\ & & 2.5 & 11.25 & 28.125 \\ \hline & 1 & 4.5 & 11.25 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 2.5 is a solution to the equation. We can use the relationships between the height and the radius to determine the height of the right circular cylinder.

$h = r + 2 = 2.5 + 2 = 4.5$

The right circular cylinder should have a radius 2.5 meters and a height 4.5 meters.

80. -

Chapter 5
Polynomial and Rational Functions
5.6 Rational Functions

Verbal

1. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?
   The rational function will be represented by a quotient of polynomial functions.
2. -
3. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?
   The numerator and denominator must have a common factor.
4. -
5. Can a graph of a rational function have no *x*-intercepts? If so, how?
   Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

Algebraic

For the following exercises, find the domain of the rational functions.

6. -
7. $f(x)=\frac{x+1}{x^2-1}$
   Begin by setting the denominator equal to zero and solving.
   $$x^2-1=0$$
   $$x^2=1$$
   $$x=\pm 1$$
   The denominator is equal to zero when $x=\pm 1$. The domain of the function is all real numbers except $x=-1,1$.
8. -
9. $f(x)=\frac{x^2+4x-3}{x^4-5x^2+4}$
   Begin by setting the denominator equal to zero and solving.
   $$x^4-5x^2+4=0$$
   $$(x^2-1)(x^2-4)=0$$
   Set each factor equal to zero.
   $(x^2-1)=0$ or $(x^2-4)=0$
   $x^2=1$ $\quad$ $x^2=4$
   $x=\pm 1$ $\quad$ $x=\pm 2$

The denominator is equal to zero when $x = \pm 1, \pm 2$. The domain of the function is all real numbers except $x = -1, 1, -2, 2$.

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

10. -

11. $f(x) = \dfrac{2}{5x+2}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$5x + 2 = 0$$
$$5x = -2$$
$$x = -\frac{2}{5}$$

The vertical asymptote is $x = -\dfrac{2}{5}$.

Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

The domain of the function is all real numbers except $x = -\dfrac{2}{5}$ because the denominator is equal to zero when $x = -\dfrac{2}{5}$.

12. -

13. $f(x) = \dfrac{x}{x^2 + 5x - 36}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x^2 + 5x - 36 = 0$$
$$(x-4)(x+9) = 0$$

Set each factor equal to zero.

$$(x-4) = 0 \quad \text{or} \quad (x+9) = 0$$
$$x = 4 \qquad\qquad x = -9$$

The vertical asymptotes are $x = 4, -9$.

Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

The domain of the function is all real numbers except $x = 4, -9$ because the denominator is equal to zero when $x = 4, -9$.

14. -

15. $f(x) = \dfrac{3x-4}{x^3-16x}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x^3 - 16x = 0$$
$$x\left(x^2 - 16\right) = 0$$
$$x\left(x+4\right)\left(x-4\right) = 0$$

Set each factor equal to zero.

$x = 0$ or $(x-4) = 0$ or $(x+4) = 0$

$x = 4$ $\quad$ $x = -4$

The vertical asymptotes are $x = 0, 4, -4$.

Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

The domain of the function is all real numbers except $x = 0, 4, -4$ because the denominator is equal to zero when $x = 0, 4, -4$.

16. -

17. $f(x) = \dfrac{x+5}{x^2-25}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x^2 - 25 = 0$$
$$x^2 = 25$$
$$x = \pm 5$$

The vertical asymptotes are $x = 5, -5$.

Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

The domain of the function is all real numbers except $x = 5, -5$ because the denominator is equal to zero when $x = 5, -5$.

18. -

19. $f(x)=\frac{4-2x}{3x-1}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$3x-1=0$$
$$3x=1$$
$$x=\frac{1}{3}$$

The vertical asymptote is $x=\frac{1}{3}$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the coefficients of the leading terms. There is a horizontal asymptote at $y=\frac{-2}{3}$ or $y=-\frac{2}{3}$.

The domain of the function is all real numbers except $x=\frac{1}{3}$ because the denominator is equal to zero when $x=\frac{1}{3}$.

For the following exercises, find the *x*- and *y*-intercepts for the functions.

20. -

21. $f(x)=\frac{x}{x^2-x}$

We can find the *y*-intercept by evaluating the function at zero.

$$f(0)=\frac{0}{0^2-0}=\text{undefined}$$

This function has no *y*-intercept.

The *x*-intercepts will occur when the function is equal to zero:

$$0=\frac{x}{x^2-x}$$

$$0=\frac{1}{x-1}$$ This is zero when the numerator is zero.

Thus, the function has no *x*-intercept.

22. -

23. $f(x)=\dfrac{x^2+x+6}{x^2-10x+24}$

We can find the *y*-intercept by evaluating the function at zero.

$$f(0)=\frac{0^2+0+6}{0^2-10(0)+24}=\frac{6}{24}=\frac{1}{4}$$

The *y*-intercept is $\left(0,\dfrac{1}{4}\right)$.

The *x*-intercepts will occur when the function is equal to zero:

$$0=\frac{x^2+x+6}{x^2-10x+24}$$ This is zero when the numerator is zero.

$$0=x^2+x+6$$

For the equation $x^2+x+6=0,$ we have $a=1, b=1,$ and $c=6.$ Substituting these values into the quadratic formula we have:

$$\begin{aligned}x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\&=\frac{-1\pm\sqrt{1^2-4\cdot1\cdot6}}{2\cdot1}\\&=\frac{-1\pm\sqrt{1-24}}{2}\\&=\frac{-1\pm i\sqrt{23}}{2}\end{aligned}$$

Since the roots are not real, the function has no *x*-intercept.

24. -

For the following exercises, describe the local and end behavior of the functions.

25. $f(x)=\dfrac{x}{2x+1}$

The given function is undefined at $x=-\dfrac{1}{2}$; so $-\dfrac{1}{2}$ is not in the domain. As the input values approach $-\dfrac{1}{2}$ from the left side, the function values increase without bound (in other words, they approach infinity).

We write in arrow notation as $x\to-\dfrac{1}{2}^-$, $f(x)\to\infty$.

As the input values approach $-\frac{1}{2}$ from the right side, the function values decrease without bound (approaching negative infinity).

We write in arrow notation as $x \to -\frac{1}{2}^{+}$, $f(x) \to -\infty$.

As the values of x approach infinity, the function values approach $\frac{1}{2}$. As the values of x approach negative infinity, the function values approach $\frac{1}{2}$.

Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to \frac{1}{2}$, and as $x \to -\infty$, $f(x) \to \frac{1}{2}$.

26. -

27. $f(x) = \frac{-2x}{x-6}$

The given function is undefined at $x = 6$; so 6 is not in the domain. As the input values approach 6 from the left side, the function values increase without bound (in other words, they approach infinity).

We write in arrow notation as $x \to 6^{-}$, $f(x) \to \infty$.

As the input values approach 6 from the right side, the function values decrease without bound (approaching negative infinity).

We write in arrow notation as $x \to 6^{+}$, $f(x) \to -\infty$.

As the values of x approach infinity, the function values approach -2. As the values of x approach negative infinity, the function values approach -2.

Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to -2$, and as $x \to -\infty$, $f(x) \to -2$.

28. -

29. $f(x) = \frac{2x^2 - 32}{6x^2 + 13x - 5}$

Factor the numerator and the denominator.

$$\begin{aligned} f(x) &= \frac{2x^2 - 32}{6x^2 + 13x - 5} \\ &= \frac{2(x-4)(x+4)}{(3x-1)(2x+5)} \end{aligned}$$

The given function is undefined at $x=\frac{1}{3}$ and $x=-\frac{5}{2}$; so $\frac{1}{3}$ and $-\frac{5}{2}$ are not in the domain. As the input values approach $-\frac{5}{2}$ from the left side or $\frac{1}{3}$ from the right side, the function values increase without bound (in other words, they approach infinity).

We write in arrow notation as $x \to -\frac{5}{2}^-$, $f(x) \to \infty$ as $x \to \frac{1}{3}^+$, $f(x) \to \infty$.

As the input values approach $-\frac{5}{2}$ from the right side or $\frac{1}{3}$ from the left side, the function values decrease without bound (approaching negative infinity).

We write in arrow notation as $x \to -\frac{5}{2}^+$, $f(x) \to -\infty$ as $x \to \frac{1}{3}^-$, $f(x) \to -\infty$.

As the values of x approach infinity, the function values approach $\frac{1}{3}$. As the values of x approach negative infinity, the function values approach $\frac{1}{3}$.

Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to \frac{1}{3}$, and as $x \to -\infty$, $f(x) \to \frac{1}{3}$.

For the following exercises, find the slant asymptote of the functions.

30. -

31. $f(x)=\dfrac{4x^2-10}{2x-4}$

Since the degree of the numerator = degree of the denominator plus 1, there is a slant asymptote found at $\frac{4x^2-10}{2x-4}$.

First, we need to change the coefficient of x in the divisor 1. Hence, we divide the numerator and denominator by 2 to get

$\frac{2x^2-5}{x-2}$.

Now, the binomial divisor is $x-2$ so $k=2$.

Add each column, multiply the result by 2, and repeat until the last column is reached.

$$\begin{array}{r|rrr} 2 & 2 & 0 & -5 \\ & & 4 & 8 \\ \hline & 2 & 4 & 3 \end{array}$$

The quotient is $2x+4$ and the remainder is 3. There is a slant asymptote at $y=2x+4$.

32. -

33. $f(x)=\dfrac{6x^3-5x}{3x^2+4}$

Since the degree of the numerator = degree of the denominator plus 1, there is a slant asymptote found at $\dfrac{6x^3-5x}{3x^2+4}$.

$$\begin{array}{r|l}
 & \quad 2x \\
3x^2+0x+4 & 6x^3+0x^2-5x \\
 & \underline{-(6x^3+0x^2+8x)} \\
 & \qquad\qquad -13x
\end{array}$$

$6x^3$ divided by $3x^2$ is $2x$.

Multiply $3x^2+0x+4$ by $2x$.

Subtract.

The quotient is $2x$ and the remainder is $-13x$. There is a slant asymptote at $y=2x$.

34. -

Graphical

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

35. The reciprocal function shifted up two units.

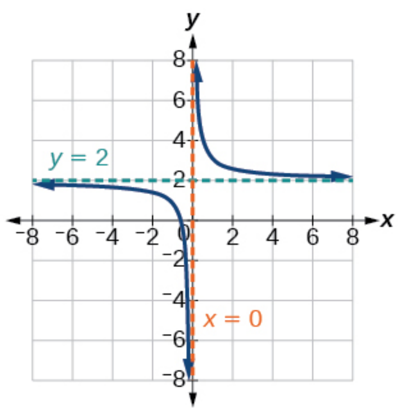
Shifting the graph up 2 would result in the function.

$$f(x)=\frac{1}{x}+2$$

or equivalently, by giving the terms a common denominator,

$$f(x)=\frac{2x+1}{x}$$

The graph of the shifted function is shown below.



Notice that this function is undefined at $x=0$, so the graph is showing a vertical asymptote at $x=0$.

As $x \to 0^-, f(x) \to -\infty$, and as $x \to 0^+, f(x) \to \infty$.

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 2, indicating a horizontal asymptote at $y = 2$.

As $x \to \pm\infty, f(x) \to 2$.

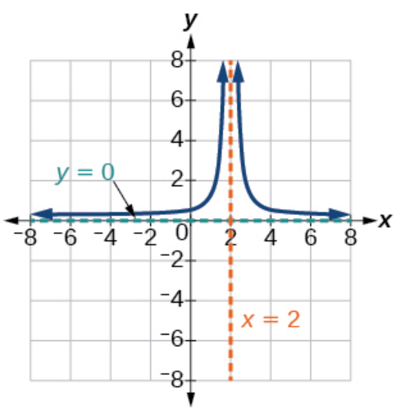
36. -

37. The reciprocal squared function shifted to the right 2 units.

Shifting the graph right 2 units would result in the function

$$f(x) = \frac{1}{(x-2)^2}$$

The graph of the shifted function is shown below.



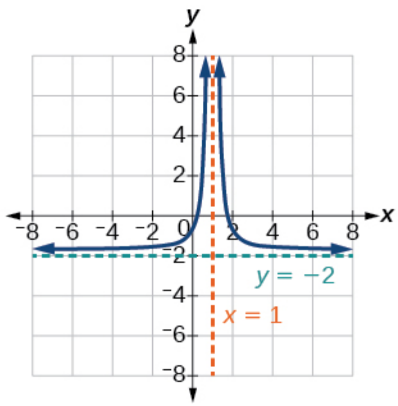
Notice that this function is undefined at $x = 2$, so the graph is showing a vertical asymptote at $x = 2$.

As $x \to 2^-, f(x) \to \infty$, and as $x \to 2^+, f(x) \to \infty$.

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 0, indicating a horizontal asymptote at $y = 0$.

As $x \to \pm\infty, f(x) \to 0$.

38. -



Notice that this function is undefined at $x=1$, so the graph is showing a vertical asymptote at $x=1$.

As $x \to 1^-$, $f(x) \to \infty$, and as $x \to 1^+$, $f(x) \to \infty$.

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of -2, indicating a horizontal asymptote at $y=-2$.

As $x \to \pm\infty$, $f(x) \to -2$.

For the following exercises, find the *x*- intercepts, the *y*- intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

39. $p(x)=\dfrac{2x-3}{x+4}$

We can find the *y*-intercept by evaluating the function at zero.

$$p(0)=\frac{2(0)-3}{(0)+4}=-\frac{3}{4}$$

The *x*-intercepts will occur when the function is equal to zero:

$$0=\frac{2x-3}{x+4} \qquad \text{This is zero when the numerator is zero.}$$

$$0=2x-3$$

$$x=\frac{3}{2}$$

The *y*-intercept is $\left(0,-\frac{3}{4}\right)$, the *x*-intercept is $\left(\frac{3}{2},0\right)$.

At the *x*-intercept, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

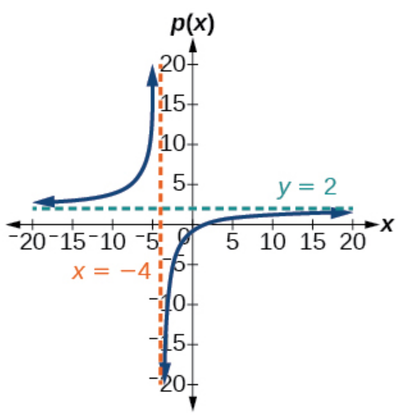
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x + 4 = 0$$
$$x = -4$$

The vertical asymptote is $x = -4$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{2}{1}$ or $y = 2$.

The graph of the given function is shown below.



40. -

41. $s(x) = \dfrac{4}{(x-2)^2}$

We can find the *y*-intercept by evaluating the function at zero.

$$s(0) = \frac{4}{(0-2)^2} = \frac{4}{4} = 1$$

The *y*-intercept is $(0,1)$.

The *x*-intercepts will occur when the function is equal to zero:

$$0 = \frac{4}{(x-2)^2}$$ This is zero when the numerator is zero.

So, the function has no *x*-intercepts.

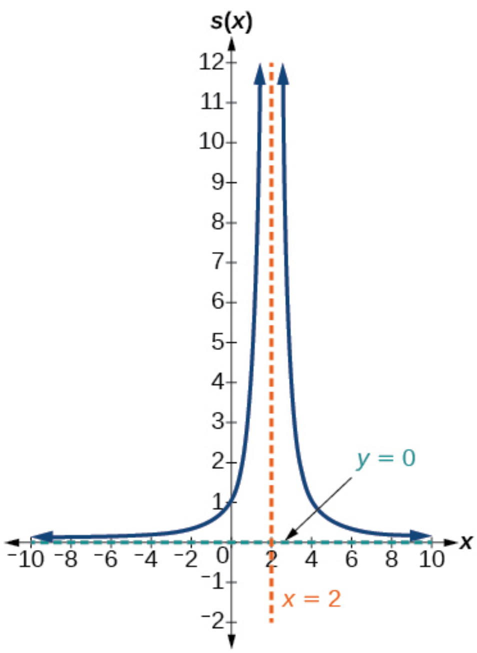
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned}(x-2)^2 &= 0\\ x-2 &= 0\\ x &= 2\end{aligned}$$

The vertical asymptote is $x = 2.$

Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0.$

The graph of the given function is shown below.



42. -

43. $f(x) = \dfrac{3x^2 - 14x - 5}{3x^2 + 8x - 16}$

First, factor the numerator and denominator.

$$\begin{aligned}f(x) &= \frac{3x^2 - 14x - 5}{3x^2 + 8x - 16}\\ &= \frac{(3x+1)(x-5)}{(3x-4)(x+4)}\end{aligned}$$

We can find the y-intercept by evaluating the function at zero.

$$f(0) = \frac{(3(0)+1)(0-5)}{(3(0)-4)(0+4)}$$
$$= \frac{(1)(-5)}{(-4)(4)}$$
$$= \frac{5}{16}$$

The *y*-intercept is $\left(0, \frac{5}{16}\right)$.

The *x*-intercepts will occur when the function is equal to zero:

$$0 = \frac{(3x+1)(x-5)}{(3x-4)(x+4)}$$ This is zero when the numerator is zero.

$$0 = (3x+1)(x-5)$$

Set each factor equal to zero.

$3x+1=0$ or $x-5=0$

$x = -\frac{1}{3}$ $x = 5$

The *x*-intercepts are $(5,0)$ and $\left(-\frac{1}{3}, 0\right)$.

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

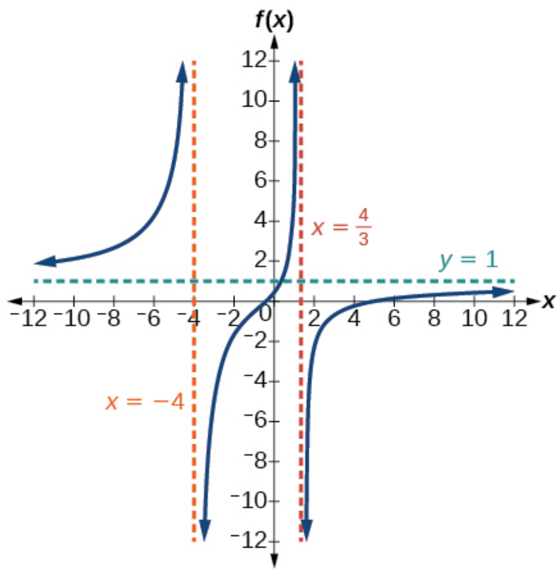
$3x-4=0$ or $x+4=0$

$x = \frac{4}{3}$ $x = -4$

The vertical asymptotes are at $x=-4$ and $x = \frac{4}{3}$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y=1$.

The graph of the given function is shown.



44. -

45. $a(x)=\dfrac{x^2+2x-3}{x^2-1}$

First, factor the numerator and denominator.

$$\begin{aligned} a(x) &= \frac{x^2+2x-3}{x^2-1} \\ &= \frac{(x-1)(x+3)}{(x+1)(x-1)} \\ &= \frac{x+3}{x+1} \qquad \text{Cancel the common factor.} \end{aligned}$$

We can find the y-intercept by evaluating the function at zero.

$$\begin{aligned} a(0) &= \frac{0+3}{0+1} \\ &= 3 \end{aligned}$$

The y-intercept is $(0,3)$.

The x-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{x+3}{x+1} \\ 0 &= x+3 \\ -3 &= x \end{aligned}$$ This is zero when the numerator is zero.

The *x*-intercept is $(-3, 0)$.

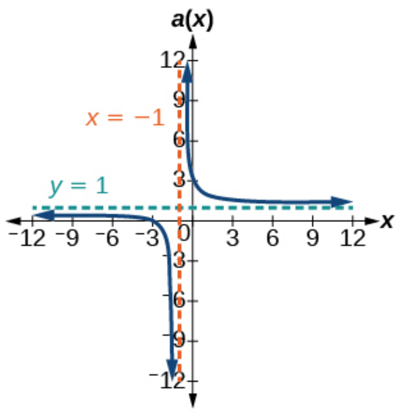
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

The vertical asymptote is at $x = -1$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{1}{1}$ or $y = 1$.

The graph of the given function is shown.



46. -

47. $h(x) = \dfrac{2x^2+x-1}{x-4}$

First, factor the numerator and denominator.

$$\begin{aligned} h(x) &= \frac{2x^2+x-1}{x-4} \\ &= \frac{(2x-1)(x+1)}{(x-4)} \end{aligned}$$

We can find the *y*-intercept by evaluating the function at zero.

$$h(0)=\frac{(2x-1)(x+1)}{(x-4)}$$
$$=\frac{(2(0)-1)(0+1)}{(0-4)}$$
$$=\frac{1}{4}$$

The *y*-intercept is $\left(0,\frac{1}{4}\right)$.

The *x*-intercepts will occur when the function is equal to zero:

$$0=\frac{(2x-1)(x+1)}{(x-4)}$$ This is zero when the numerator is zero.

$$0=(2x-1)(x+1)$$

Set each factor equal to zero.

$2x-1=0$ or $x+1=0$

$x=\frac{1}{2}$ $\qquad x=-1$

The *x*-intercepts are $(-1,0)$ and $\left(\frac{1}{2},0\right)$.

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x-4=0$$
$$x=4$$

The vertical asymptote is at $x=4$.

Since the degree of the numerator = degree of the denominator plus 1, there is a slant asymptote.

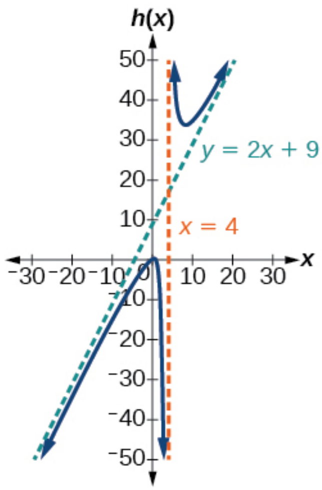
Now, the binomial divisor is $x-4$ so $k=4$.

Add each column, multiply the result by 4, and repeat until the last column is reached.

$$\begin{array}{r|rrr} 4 & 2 & 1 & -1 \\ & & 8 & 36 \\ \hline & 2 & 9 & 35 \end{array}$$

The quotient is $2x+9$ and the remainder is 35. There is a slant asymptote at $y=2x+9$.

The graph of the given function is shown below.



48. -

49. $w(x)=\dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

We can find the *y*-intercept by evaluating the function at zero.

$$w(0)=\frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$$
$$=\frac{(0-1)(0+3)(0-5)}{(0+2)^2(0-4)}$$
$$=-\frac{15}{16}$$

The *y*-intercept is $\left(0,-\dfrac{15}{16}\right)$.

The *x*-intercepts will occur when the function is equal to zero:

$$0=\frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$$ This is zero when the numerator is zero.

$$0=(x-1)(x+3)(x-5)$$

Set each factor equal to zero.

$x-1=0$ or $x+3=0$ or $x-5=0$

$x=1$ $\quad x=-3$ $\quad x=5$

The *x*-intercepts are $(1,0)$, $(-3,0)$ and $(5,0)$.

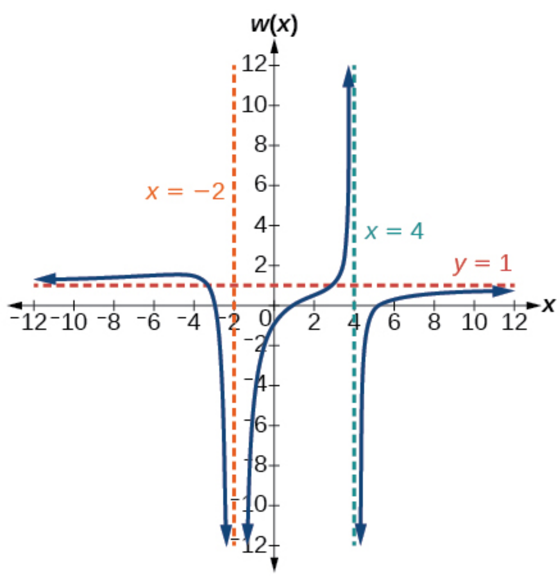
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} (x+2)^2 &= 0 \quad \text{or} \quad x-4=0 \\ x+2 &= 0 \quad \text{or} \quad x=4 \\ x &= -2 \end{aligned}$$

The vertical asymptotes are at $x=-2$ and $x=4$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y=\frac{1}{1}$ or $y=1$.

The graph of the given function is shown.



50. -

For the following exercises, write an equation for a rational function with the given characteristics.

51. Vertical asymptotes at $x=5$ and $x=-5$

x-intercepts at $(2,0)$ and $(-1,0)$ y-intercept at $(0,4)$

Use the given information to write a function of the form

$$f(x)=a\frac{(x-2)(x+1)}{(x+5)(x-5)}$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0,4)$.

$$4 = a\frac{(0-2)(0+1)}{(0+5)(0-5)}$$

$$4 = a\left(\frac{-2}{-25}\right)$$

$$a = \frac{4\cdot 25}{2} = 50$$

This gives us a final function of $f(x) = 50\frac{(x-2)(x+1)}{(x+5)(x-5)} = 50\frac{x^2-x-2}{x^2-25}$.

52. -

53. Vertical asymptotes at $x=-4$ and $x=-5$

x-intercepts at $(4,0)$ and $(-6,0)$ Horizontal asymptote at $y=7$

Use the given information to write a function of the form

$$f(x) = a\frac{(x-4)(x+6)}{(x+4)(x+5)}$$

Remember that the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.
Hence, $a=7$.

This gives us a final function of $f(x) = 7\frac{(x-4)(x+6)}{(x+4)(x+5)} = 7\frac{x^2+2x-24}{x^2+9x+20}$.

54. -

55. Vertical asymptote at $x=-1$
Double zero at $x=2$ y-intercept at $(0,2)$
Use the given information to write a function of the form

$$f(x) = a\frac{(x-2)^2}{(x+1)}$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0,2)$.

$$2 = a\frac{(0-2)^2}{(0+1)}$$

$$2 = a\left(\frac{4}{1}\right)$$

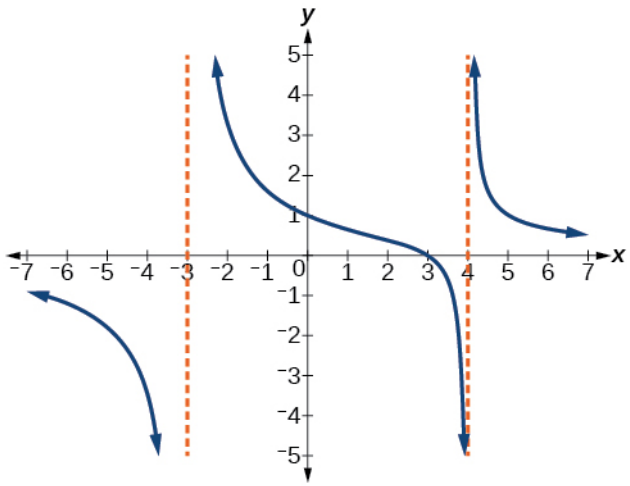
$$a = \frac{1}{2}$$

This gives us a final function of $f(x) = \frac{1}{2} \cdot \frac{(x-2)^2}{(x+1)} = \frac{1}{2} \cdot \frac{x^2 - 4x + 4}{(x+1)}$.

56. -

For the following exercises, use the graphs to write an equation for the function.

57.



The graph appears to have an *x*-intercept at $x = 3$. At this point, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes, $x = -3$ and $x = 4$. Both asymptotes are exhibiting a behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other.
We can use this information to write a function of the form

$$f(x) = a\frac{(x-3)}{(x+3)(x-4)}$$

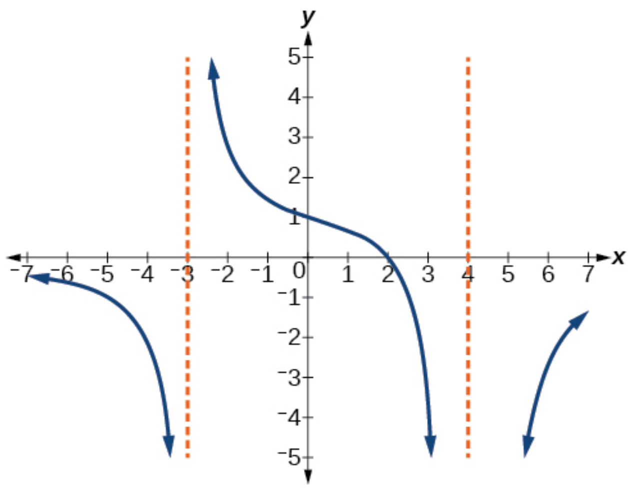
To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,1)$.

$$1 = a\frac{(0-3)}{(0+3)(0-4)}$$
$$1 = a\left(\frac{-3}{-12}\right)$$
$$a = 4$$

This gives us a final function of $f(x) = \dfrac{4(x-3)}{(x+3)(x-4)} = \dfrac{4(x-3)}{x^2-x-12}$.

58. -

59.



The first asymptote exhibits behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other, while the second one exhibits behavior like $-\frac{1}{x^2}$, with the function decreasing without bound on either side.

$$f(x) = a\frac{(x-2)}{(x+3)(x-4)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,1)$.

$$1 = a\frac{(0-2)}{(0+3)(0-4)^2}$$

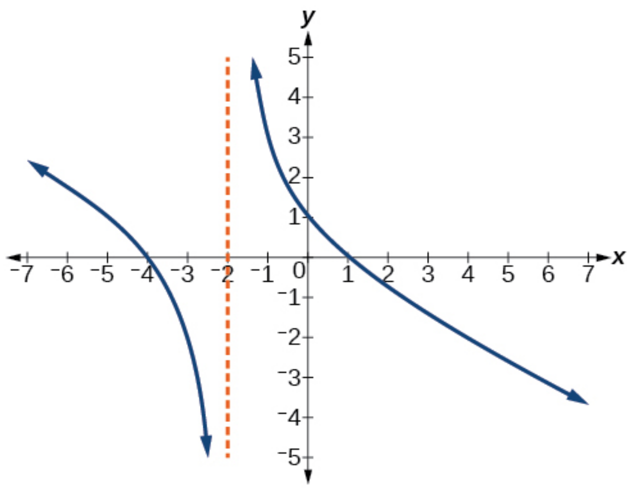
$$1 = a\left(\frac{-2}{48}\right)$$

$$a = -24$$

This gives us a final function of $f(x) = -24\frac{(x-2)}{(x+3)(x-4)^2}$.

60. -

61.



The graph appears to have *x*-intercepts at $x = -4$ and $x = 1$. At both, the graph passes through the intercept, suggesting linear factors. The graph has a vertical asymptote at $x = -2$.

It seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+4)(x-1)}{(x+2)}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,1)$.

$$1 = a\frac{(0+4)(0-1)}{(0+2)}$$

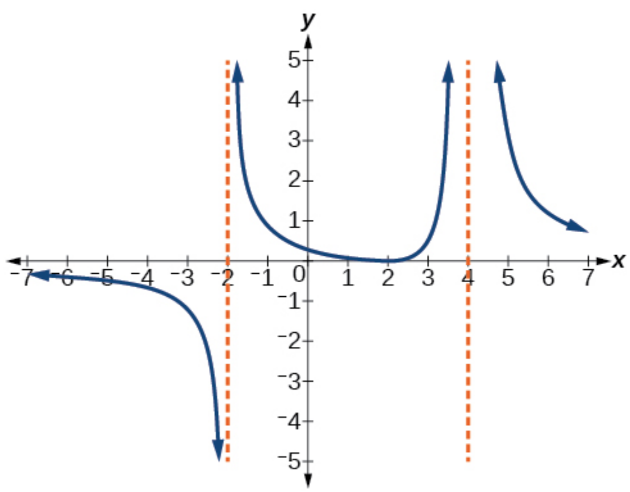
$$1 = a\left(\frac{-4}{2}\right)$$

$$a = -\frac{1}{2}$$

This gives us a final function of $f(x) = -\frac{1}{2}\cdot\frac{(x+4)(x-1)}{(x+2)} = -\frac{1}{2}\cdot\frac{\left(x^2+3x-4\right)}{(x+2)}$.

62. -

63.



The graph appears to have *x*-intercept at $x = 2$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2). The graph has two vertical asymptotes. The one at $x = -2$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 4$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward infinity on both sides of the asymptote.

We can use this information to write a function of the form

$$f(x) = a\frac{(x-2)^2}{(x+2)(x-4)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $\left(0, \frac{1}{4}\right)$.

$$\frac{1}{4} = a\frac{(0-2)^2}{(0+2)(0-4)^2}$$

$$\frac{1}{4} = a\left(\frac{4}{2\cdot 16}\right)$$

$$a = \frac{32}{16} = 2$$

This gives us a final function of $f(x) = \dfrac{2(x-2)^2}{(x+2)(x-4)^2}$.

Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote.

64. -

65. $f(x) = \dfrac{x}{x-3}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x - 3 = 0$$
$$x = 3$$

The vertical asymptote is $x = 3$.

Since the degree of the denominator = degree of the numerator so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{1}{1}$ or $y = 1$.

As the input values approach 3 from the left side, the function values decrease without bound (in other words, they approach negative infinity). As the input values approach 2 from the right side, the function values increase without bound (approaching infinity). We can see this behavior in the table.

x	3.1	3.01	3.001	2.99	2.999
y	31	301	3,001	–299	–2,999

We write in arrow notation

As $x \to 3^-$, $f(x) \to -\infty$, and as $x \to 3^+$, $f(x) \to \infty$.

As the values of x approach infinity, the function values approach 1. As the values of x approach negative infinity, the function values approach 1. We can see this behavior in the table.

x	10	100	1,000	10,000	100,000
y	1.4286	1.0309	1.003	1.0003	1.00003

We write in arrow notation

As $x \to \infty,\ f(x) \to 1$, and as $x \to -\infty,\ f(x) \to 1$.

66. -

67. $f(x) = \dfrac{2x}{(x-3)^2}$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned}(x-3)^2 &= 0\\ x-3 &= 0\\ x &= 3\end{aligned}$$

The vertical asymptote is $x = 3$.

Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

As the input values approach 3 from the left side, the function values increase without bound (in other words, they approach infinity). As the input values approach 3 from the right side, the function values increase without bound (approaching infinity).

We can see this behavior in the table.

x	3.1	3.01		3.001	2.9	2.99
y	620	60,200		600,200	580	59,800

We write in arrow notation

As $x \to 3^-,\ f(x) \to \infty$, and as $x \to 3^+,\ f(x) \to \infty$.

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. We can see this behavior in the table.

x	10	100	1,000	10,000	100,000
y	.40816	.02126	.00201	.0002	.00002

We write in arrow notation

As $x \to \infty,\ f(x) \to 0$, and as $x \to -\infty,\ f(x) \to 0$.

68. –

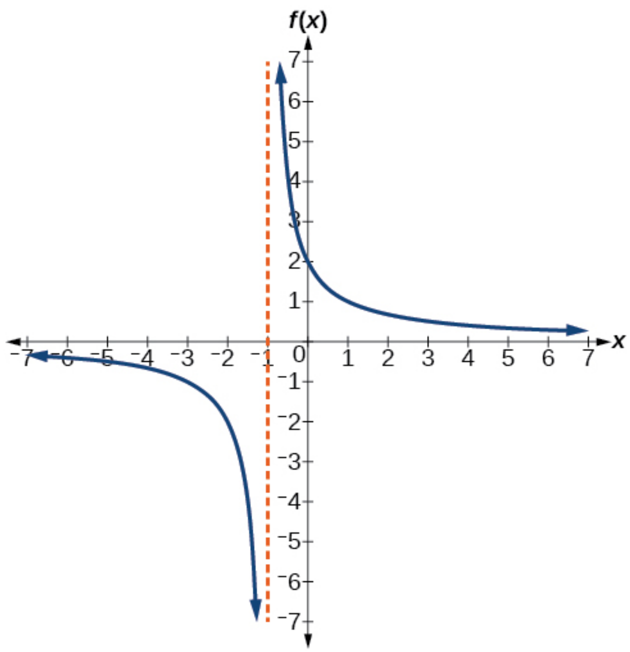
Technology

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

69. $f(x) = \dfrac{2}{x+1}$

Set $\boxed{Y1=}\ \boxed{2}\ \boxed{/}\ \boxed{(}\ \boxed{x}\ \boxed{+}\ \boxed{1}\ \boxed{)}$

A graph of this function, as shown below, confirms that the function is not defined when $x = -1$.



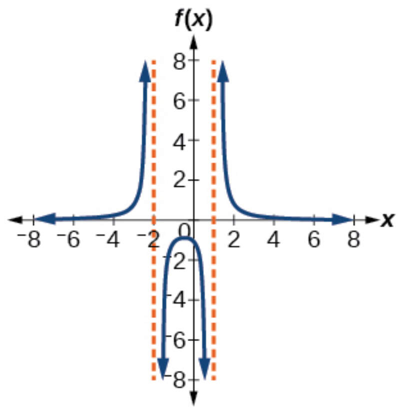
From the graph, $f(x) > 0$ when $x \in (-1, \infty)$.

70. -

71. $f(x) = \dfrac{2}{(x-1)(x+2)}$

Set $\boxed{Y1=}\ \boxed{2}\ \boxed{/}\ \boxed{(}\boxed{(}\boxed{x}\ \boxed{-}\ \boxed{1}\ \boxed{)}\boxed{(}\boxed{x}\ \boxed{+}\ \boxed{2}\ \boxed{)}\boxed{)}$

A graph of this function, as shown below, confirms that the function is not defined when $x = -2$ and $x = 1$.



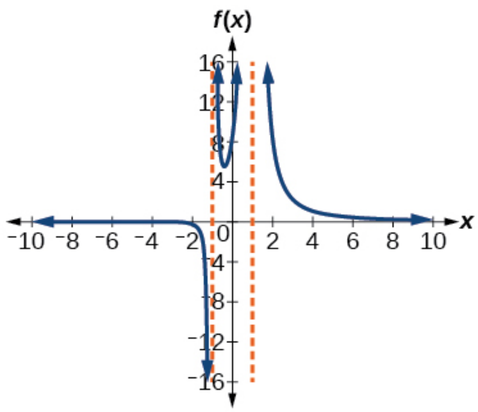
From the graph, $f(x) > 0$ when $x \in (-\infty, -2) \cup (1, \infty)$.

72. -

73. $f(x) = \dfrac{(x+3)^2}{(x-1)^2(x+1)}$

Set $\boxed{Y1=}\ \boxed{(}\boxed{x}\ \boxed{+}\ \boxed{3}\boxed{)}\boxed{\wedge}\boxed{2}\ \boxed{/}\ \boxed{(}\boxed{(}\boxed{x}\ \boxed{-}\ \boxed{1}\boxed{)}\boxed{\wedge}\boxed{2}\boxed{(}\boxed{x}\ \boxed{+}\ \boxed{1}\boxed{)}\boxed{)}$

A graph of this function, as shown below, confirms that the function is not defined when $x = 1$ and $x = -1$.



From the graph, $f(x) > 0$ when $x \in (-1, 1) \cup (1, \infty)$.

Extensions

For the following exercises, identify the removable discontinuity.

74. -

75. $f(x) = \dfrac{x^3+1}{x+1}$

Factor the numerator and the denominator.

$$f(x)=\frac{(x+1)(x^2-x+1)}{x+1}$$

Notice that there is a common factor in the numerator and the denominator, $x+1$. The zero for this factor is $x=-1$. This is the location of the removable discontinuity.

$$f(-1)=(-1)^2-(-1)+1=3$$

The removable discontinuity is at $(-1,3)$.

76. -

77. $f(x)=\dfrac{2x^2+5x-3}{x+3}$

Factor the numerator and the denominator.

$$f(x)=\frac{(2x-1)(x+3)}{x+3}$$

Notice that there is a common factor in the numerator and the denominator, $x+3$. The zero for this factor is $x=-3$. This is the location of the removable discontinuity.

$$f(-3)=2(-3)-1=-7$$

The removable discontinuity is at $(-3,-7)$.

78. -

Real-World Applications

For the following exercises, express a rational function that describes the situation.

79. A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 3 pounds per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

water: $W(t)=200+10t$ in gallons

sugar: $S(t)=10+3t$ in pounds

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{10+3t}{200+10t}$$

80. -

For the following exercises, use the given rational function to answer the question.

81. The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C(t)=\frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?

 Remember that a horizontal asymptote of a function is a horizontal line where the graph approaches the line as the inputs increase or decrease without bound.
 Since the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $C(t)=0$, that is as t increases, C approaches 0.
 We write in arrow notation
 as $t \to \infty,\ C(t) \to 0.$

82. -

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

83. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.

 Let the height of the box be *h*, then the volume of the box is area of the base $\times$ height, that is $V = x^2h$ as the base is the square.
 Substitute the value of volume to express *h* in terms *x*.

$$108 = x^2h$$

$$h=\frac{108}{x^2}$$

 The surface area of the given open box is $A(x)=x^2+4xh.$
 Substitute the expression of h in $A(x).$

$$A(x)=x^2+4x\left(\frac{108}{x^2}\right)$$
$$=x^2+\frac{432}{x}$$

 Set $\boxed{Y1=}$ $\boxed{x}$ $\boxed{\wedge}$ $\boxed{2}$ $\boxed{+}$ $\boxed{432}$ $\boxed{/}$ $\boxed{x}$
 Use the calculator's Minimum feature to approximate the coordinates of the local minimum shown on the graph.

Open the Graph screen's CALCULATE menu by pressing $\boxed{2^{nd}}$ $\boxed{\text{CALC}}$.
Select 3: minimum.
The calculator returns to the graph and prompts you for a left bound.
Move the cursor until it is slightly to the left of the minimum. Press $\boxed{\text{ENTER}}$.
Move the cursor until it is slightly to the right of the minimum. Press $\boxed{\text{ENTER}}$.
The calculator prompts you for an initial guess for the minimum. Move the cursor close to the minimum. Press $\boxed{\text{ENTER}}$.
The calculator improves on the initial guess and returns an approximation of the coordinates of the local minimum point: $(6, 108)$

Substitute the value of x in $h=\dfrac{108}{x^2}$.

$$h=\frac{108}{6^2}=\frac{108}{36}=3$$

The dimension of the given box is 6 by 6 by 3 inches.

84. -
85. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.
A right circular cylinder with radius x units and height h units has a volume of V cubic units given by $V=\pi x^2 h$.
Substitute the value of volume to express h in terms x.

$$100=\pi x^2 h$$
$$h=\frac{100}{\pi x^2}$$

The surface area of the given right circular cylinder is $A(x)=2\pi x^2+2\pi xh$.
Substitute the expression of h in $A(x)$.

$$A(x)=2\pi x^2+2\pi x\left(\frac{100}{\pi x^2}\right)$$
$$=2\pi x^2+\frac{200}{x}$$

Set $\boxed{Y1=}$ $\boxed{2}\boxed{\pi}\boxed{x}\boxed{\wedge}\boxed{2}$ $\boxed{+}$ $\boxed{200}$ $\boxed{/}$ $\boxed{x}$
Use the calculator's Minimum feature to approximate the coordinates of the local minimum shown on the graph.
Open the Graph screen's CALCULATE menu by pressing $\boxed{2^{nd}}$ $\boxed{\text{CALC}}$.
Select 3: minimum.

The calculator returns to the graph and prompts you for a left bound.

Move the cursor until it is slightly to the left of the minimum. Press $\boxed{\text{ENTER}}$.

Move the cursor until it is slightly to the right of the minimum. Press $\boxed{\text{ENTER}}$.

The calculator prompts you for an initial guess for the minimum. Move the cursor close to the minimum. Press $\boxed{\text{ENTER}}$.

The calculator improves on the initial guess and returns an approximation of the coordinates of the local minimum point: $(2.52, 119.27)$

Substitute the value of x in $h = \dfrac{100}{\pi x^2}$.

$$h = \frac{100}{\pi(2.52)^2} \approx 5.03$$

The right circular cylinder should have a radius 2.52 inches and a height 5.03 inches.

86. -

87. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents /square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

A right circular cylinder with radius x units and height h units has a volume of V cubic units given by $V = \pi x^2 h$.

Substitute the value of volume to express h in terms x.

$$40 = \pi x^2 h$$

$$h = \frac{40}{\pi x^2}$$

Given that it costs 4 cents /square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder.

The surface area of the given right circular cylinder is

$$S.A \ \cos t = \text{Top area cost + Base area cost + rest of the area in cylinder}$$

$$\begin{aligned} A(x) &= \pi x^2(4) + \pi x^2(4) + 2\pi x h(1) \\ &= 8\pi x^2 + 2\pi x h \end{aligned}$$

Substitute the expression of h in $A(x)$.

$$\begin{aligned} A(x) &= 8\pi x^2 + 2\pi x\left(\frac{40}{\pi x^2}\right) \\ &= 8\pi x^2 + \frac{80}{x} \end{aligned}$$

Set $\boxed{Y1=}$ $\boxed{8}\boxed{\pi}\boxed{x}\boxed{\wedge}\boxed{2}$ $\boxed{+}$ $\boxed{80}$ $\boxed{/}$ $\boxed{x}$

Use the calculator's Minimum feature to approximate the coordinates of the local minimum shown on the graph.

Open the Graph screen's CALCULATE menu by pressing $\boxed{2^{\text{nd}}}$ $\boxed{\text{CALC}}$.

Select 3: minimum.

The calculator returns to the graph and prompts you for a left bound.

Move the cursor until it is slightly to the left of the minimum. Press $\boxed{\text{ENTER}}$.

Move the cursor until it is slightly to the right of the minimum. Press $\boxed{\text{ENTER}}$.

The calculator prompts you for an initial guess for the minimum. Move the cursor close to the minimum. Press $\boxed{\text{ENTER}}$.

The calculator improves on the initial guess and returns an approximation of the coordinates of the local minimum point: $(1.17, 102.78)$

Thus, the right circular cylinder should have a radius 1.17 inches.

Chapter 5

Polynomial and Rational Functions

5.7 Inverses and Radical Functions

Verbal

1. Explain why we cannot find inverse functions for all polynomial functions.

   All polynomial functions are not one-to-one. Even with restricted domain, so that it is one-to-one, it can be too difficult or impossible to solve for *x* in terms of *y*.
2. -
3. When finding the inverse of a radical function, what restriction will we need to make?

   We will need a restriction on the domain of the answer.
4. -

Algebraic

For the following exercises, find the inverse of the function on the given domain.

5. $f(x) = (x-4)^2,\ [4, \infty)$

   The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ on which it is one-to-one.

   To find the inverse, start by replacing $f(x)$ with the simple variable y.

   $y = (x-4)^2$ Interchange x and y.

   $x = (y-4)^2$ Take the square root.

   $\pm\sqrt{x} = y-4$ Add 4 to both sides.

   $4 \pm \sqrt{x} = y$

   The domain of the original function was restricted to $x \geq 4,$ so the outputs of the inverse need to be the same, $f(x) \geq 4,$ and we must use the + case:

   $$f^{-1}(x) = \sqrt{x} + 4$$
6. -

7. $f(x)=(x+1)^2-3,\ [-1,\infty)$

The original function $f(x)=(x+1)^2-3$ is not one-to-one, but the function is restricted to a domain of $x\geq -1$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=(x+1)^2-3$ Interchange x and y.

$x=(y+1)^2-3$ Add 3 to both sides.

$x+3=(y+1)^2$ Take the square root.

$\pm\sqrt{x+3}=y+1$ Subtract 1 from both sides.

$\pm\sqrt{x+3}-1=y$

The domain of the original function was restricted to $x\geq -1$, so the outputs of the inverse need to be the same, $f^{-1}(x)\geq -1$, and we must use the + case:

$$f^{-1}(x)=\sqrt{x+3}-1$$

8. -

9. $f(x)=12-x^2,\ [0,\infty)$

The original function $f(x)=12-x^2$ is not one-to-one, but the function is restricted to a domain of $x\geq 0$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=12-x^2$ Interchange x and y.

$x=12-y^2$ Move each variable onto the opposite side.

$y^2=12-x$ Take the square root.

$y=\pm\sqrt{12-x}$

The domain of the original function was restricted to $x\geq 0$, so the outputs of the inverse need to be the same, $f^{-1}(x)\geq 0$, and we must use the + case:

$$f^{-1}(x)=\sqrt{12-x}$$

10. -

11. $f(x)=2x^2+4, [0,\infty)$

The original function $f(x)=2x^2+4$ is not one-to-one, but the function is restricted to a domain of $x\geq 0$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=2x^2+4$ Interchange x and y.

$x=2y^2+4$ Subtract 4 from both sides.

$x-4=2y^2$ Divide both sides by 2.

$\frac{x-4}{2}=y^2$ Take the square root.

$\pm\sqrt{\frac{x-4}{2}}=y$

The domain of the original function was restricted to $x\geq 0$, so the outputs of the inverse need to be the same, $f^{-1}(x)\geq 0$, and we must use the $+$ case:

$$f^{-1}(x)=\sqrt{\frac{x-4}{2}}$$

For the following exercises, find the inverse of the functions.

12. -

13. $f(x)=3x^3+1$

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = 3x^3 + 1$ Interchange x and y.

$x = 3y^3 + 1$ Subtract 1 from both sides.

$x - 1 = 3y^3$ Divide both sides by 3.

$\frac{x-1}{3} = y^3$ Take cubic root on both sides.

$\sqrt[3]{\frac{x-1}{3}} = y$ Rename the function $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{\frac{x-1}{3}}$

14. -

15. $f(x) = 4 - 2x^3$

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = 4 - 2x^3$ Interchange x and y.

$x = 4 - 2y^3$ Subtract 4 from both sides.

$x - 4 = -2y^3$ Divide both sides by -2.

$\frac{4-x}{2} = y^3$ Take cubic root on both sides.

$\sqrt[3]{\frac{4-x}{2}} = y$ Rename the function $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{\frac{4-x}{2}}$

For the following exercises, find the inverse of the functions.

16. -

17. $f(x) = \sqrt{3-4x}$

The original function $f(x) = \sqrt{3-4x}$ is restricted to a domain of $x \le \frac{3}{4}$ on which it is one-to-one, because the radicand must be non-negative. It is one-to-one on its domain.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = \sqrt{3-4x}$ Interchange x and y.

$x = \sqrt{3-4y}$ Square each side.

$x^2 = 3-4y$ Subtract 1 from both sides.

$x^2 - 3 = -4y$ Divide both sides by -2.

$\dfrac{3-x^2}{4} = y$

The range of the original function was $f(x) \geq 0$, so the domain of the inverse need to be the same, $x \geq 0$.

$f^{-1}(x) = \dfrac{3-x^2}{4}; x \geq 0.$

18. -

19. $f(x) = \sqrt{6x-8} + 5$

The original function $f(x) = \sqrt{6x-8} + 5$ is one-to-one, on its domain of $x \geq \dfrac{8}{6}$.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = \sqrt{6x-8} + 5$ Interchange x and y.

$x = \sqrt{6y-8} + 5$ Subtract 5 from both sides.

$x - 5 = \sqrt{6y-8}$ Square each side.

$(x-5)^2 = 6y - 8$ Add 8 to both sides.

$(x-5)^2 + 8 = 6y$ Divide both sides by 6.

$\dfrac{(x-5)^2 + 8}{6} = y$ Rename the function $f^{-1}(x)$.

$f^{-1}(x) = \dfrac{(x-5)^2 + 8}{6}$

Note that the original function has range $f(x) \geq 5$.

domain of f^{-1} = range of $f = [5, \infty)$.

20. -

21. $f(x) = 3 - \sqrt[3]{x}$

This is a transformation of the basic cubic root function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = 3 - \sqrt[3]{x}$	Interchange x and y.
$x = 3 - \sqrt[3]{y}$	Subtract 3 from both sides.
$x - 3 = -\sqrt[3]{y}$	Divide both sides by -1.
$3 - x = \sqrt[3]{y}$	Raise a power of 3 on both sides.
$(3 - x)^3 = y$	Rename the function $f^{-1}(x)$.
$f^{-1}(x) = (3 - x)^3$	

22. -

23. $f(x) = \dfrac{3}{x - 4}$

This is a transformation of the basic reciprocal toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = \dfrac{3}{x - 4}$	Interchange x and y.
$x = \dfrac{3}{y - 4}$	Multiply $y - 4$ on both sides.
$x(y - 4) = 3$	Distribute x.
$xy - 4x = 3$	Add $4x$ to both sides.
$xy = 4x + 3$	Divide x by both sides.
$y = \dfrac{4x + 3}{x}$	Rename the function $f^{-1}(x)$.
$f^{-1}(x) = \dfrac{4x + 3}{x}$	

24. -

25. $f\left(x\right)=\dfrac{x-2}{x+7}$

This is a transformation of the basic rational function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f\left(x\right)$ with the simple variable y.

$y=\dfrac{x-2}{x+7}$	Interchange x and y.
$x=\dfrac{y-2}{y+7}$	Multiply $y+7$ on both sides.
$x\left(y+7\right)=y-2$	Distribute x.
$xy+7x=y-2$	Add 2 to both sides.
$xy+7x+2=y$	Subtract xy from both sides.
$7x+2=y-xy$	Divide $1-x$ by both sides.
$\dfrac{7x+2}{1-x}=y$	Rename the function $f^{-1}\left(x\right)$.
$f^{-1}\left(x\right)=\dfrac{7x+2}{1-x}$	

26. -

27. $f\left(x\right)=\dfrac{5x+1}{2-5x}$

This is a transformation of the basic rational function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f\left(x\right)$ with the simple variable y.

$y = \dfrac{5x+1}{2-5x}$ Interchange x and y.

$x = \dfrac{5y+1}{2-5y}$ Multiply $2-5y$ on both sides.

$x(2-5y) = 5y+1$ Distribute x.

$2x - 5xy = 5y+1$ Add $5xy$ to both sides.

$2x = 5xy + 5y + 1$ Subtract 1 from both sides.

$2x - 1 = 5xy + 5y$ Divide $5x+5$ by both sides.

$\dfrac{2x-1}{5x+5} = y$ Rename the function $f^{-1}(x)$.

$f^{-1}(x) = \dfrac{2x-1}{5x+5}$

28. -

29. $f(x) = x^2 + 4x + 1,\ [-2, \infty)$

The original function $f(x) = x^2 + 4x + 1$ is not one-to-one, but the function is restricted to a domain of $x \geq -2$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = x^2 + 4x + 1$ Interchange x and y.

$x = y^2 + 4y + 1$ Subtract x from both sides.

$0 = y^2 + 4y + 1 - x$

Use the quadratic formula to solve for y. We have $a = 1, b = 4,$ and $c = 1 - x.$

$$
\begin{aligned}
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (1-x)}}{2 \cdot 1} \\
&= \frac{-4 \pm \sqrt{16 + 4x - 4}}{2} \\
&= \frac{-4 \pm 2\sqrt{x+3}}{2} \\
&= -2 \pm \sqrt{x+3}
\end{aligned}
$$

The domain of the original function was restricted to $x \geq -2,$ so the outputs of the inverse need to be the same, $f^{-1}(x) \geq -2,$ and we must use the + case:

$f^{-1}(x) = \sqrt{x+3} - 2$

30. -

Graphical

For the following exercises, find the inverse of the function and graph both the function and its inverse.

31. $f(x) = x^2 + 2, x \geq 0$

The original function $f(x) = x^2 + 2$ is not one-to-one, but the function is restricted to a domain of $x \geq 0$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = x^2 + 2$	Interchange x and y.
$x = y^2 + 2$	Subtract 2 from both sides.
$x - 2 = y^2$	Take the square root.
$\sqrt{x-2} = y$	Rename the function $f^{-1}(x)$.
$f^{-1}(x) = \sqrt{x-2}$	

Note that the original function has range $f(x) \geq 2.$

domain of $f^{-1} =$ range of $f = [2, \infty).$

Restrict the domain of a function displayed $Y =$, divide by the restricted domain.

Key $Y1 = (x^2 + 2) \div (x \geq 0)$

This works because $\geq$ is a logical operator, which equals 1 if the statement is true and 0 if it is false. Thus, outside the domain, we get division by zero, so the function is undefined there.

To draw the inverse of that function:

[2nd PRGM makes DRAW]

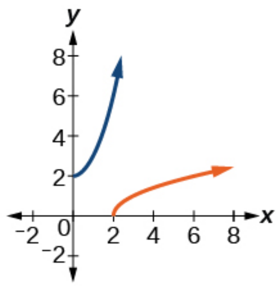
Either move the cursor down to 8 and press [Enter], or simply press [8].

Tell the TI-83/84 to find the original function in Y1.

At this point the screen shows this command: DrawInv Y1

Now execute the command, by pressing [Enter].

The result is shown.



32. -

33. $f(x)=(x+3)^2,\ x\ge -3$

The original function $f(x)=(x+3)^2$ is not one-to-one, but the function is restricted to a domain of $x\ge -3$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=(x+3)^2$	Interchange x and y.
$x=(y+3)^2$	Take the square root.
$\pm\sqrt{x}=y+3$	Subtract 3 from both sides.
$\pm\sqrt{x}-3=y$	Rename the function $f^{-1}(x)$.
$f^{-1}(x)=\sqrt{x}-3$	

Note that the original function has range $f(x)\ge 0$.

domain of $f^{-1}=$ range of $f=[0,\infty)$.

In TI calculator, restrict the domain of a function displayed $Y=$, divide by the restricted domain.

Key $Y1=(x+3)^2\div(x\ge -3)$

This works because $\ge$ is a logical operator, which equals 1 if the statement is true and 0 if it is false. Thus, outside the domain, we get division by zero, so the function is undefined there.

To draw the inverse of that function:

[2nd PRGM makes DRAW]

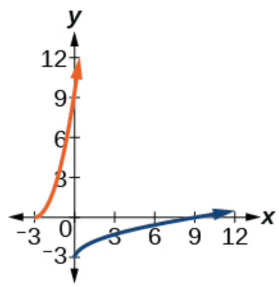
Either move the cursor down to 8 and press [Enter], or simply press [8].

Tell the TI-83/84 to find the original function in Y1.

At this point the screen shows this command: DrawInv Y1

Now execute the command, by pressing [Enter].

The result is shown.



34. -

35. $f(x)=x^3+3$

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=x^3+3$	Interchange x and y.
$x=y^3+3$	Subtract 3 from both sides.
$x-3=y^3$	Take cubic root on both sides.
$\sqrt[3]{x-3}=y$	Rename the function $f^{-1}(x)$.
$f^{-1}(x)=\sqrt[3]{x-3}$	

In TI calculator, key $Y1=x^3+3$

To draw the inverse of that function:

[2nd PRGM makes DRAW]

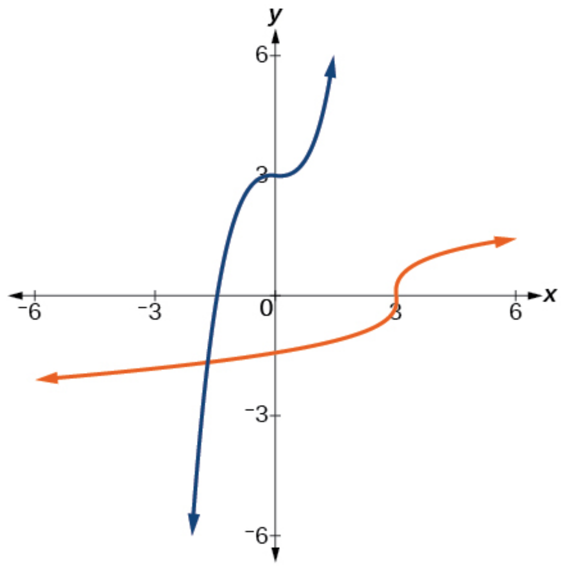
Either move the cursor down to 8 and press [Enter], or simply press [8].

Tell the TI-83/84 to find the original function in Y1.

At this point the screen shows this command: DrawInv Y1

Now execute the command, by pressing [Enter].

The result is shown.



36. -

37. $f(x) = x^2 + 4x,\ x \geq -2$

The original function $f(x) = x^2 + 4x$ is not one-to-one, but the function is restricted to a domain of $x \geq -2$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = x^2 + 4x$ Interchange x and y.

$x = y^2 + 4y$ Subtract x from both sides.

$0 = y^2 + 4y - x$

Use the quadratic formula to solve for y. We have $a = 1, b = 4,$ and $c = -x.$

$$\begin{aligned}
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot(-x)}}{2\cdot 1} \\
&= \frac{-4 \pm \sqrt{16 + 4x}}{2} \\
&= \frac{-4 \pm 2\sqrt{x+4}}{2} \\
&= -2 \pm \sqrt{x+4}
\end{aligned}$$

The domain of the original function was restricted to $x \geq -2,$ so the outputs of the inverse need to be the same, $f^{-1}(x) \geq -2,$ and we must use the + case:

$$f^{-1}(x) = \sqrt{x+4} - 2$$

In TI calculator, restrict the domain of a function displayed $Y =$, divide by the restricted domain.

Key $Y1 = x^2 + 4x \div (x \geq -2)$

This works because $\geq$ is a logical operator, which equals 1 if the statement is true and 0 if it is false. Thus, outside the domain, we get division by zero, so the function is undefined there.

To draw the inverse of that function:

[2nd PRGM makes DRAW]

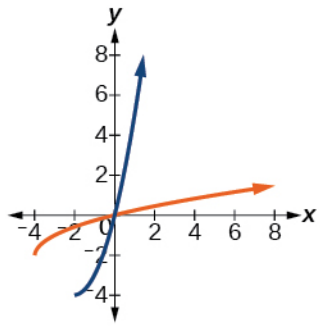
Either move the cursor down to 8 and press [Enter], or simply press [8].

Tell the TI-83/84 to find the original function in Y1.

At this point the screen shows this command: DrawInv Y1

Now execute the command, by pressing [Enter].

The result is shown.



38. –

39. $f(x)=\frac{2}{x}$

This is a transformation of the basic reciprocal toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=\frac{2}{x}$ Interchange x and y.

$x=\frac{2}{y}$

$y=\frac{2}{x}$

In TI calculator, key $Y1=\frac{2}{x}$

To draw the inverse of that function:

[2nd PRGM makes DRAW]

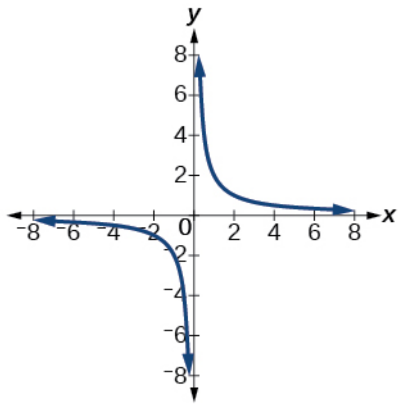
Either move the cursor down to 8 and press [Enter], or simply press [8].

Tell the TI-83/84 to find the original function in Y1.

At this point the screen shows this command: DrawInv Y1

Now execute the command, by pressing [Enter].

The result is shown.



40. -

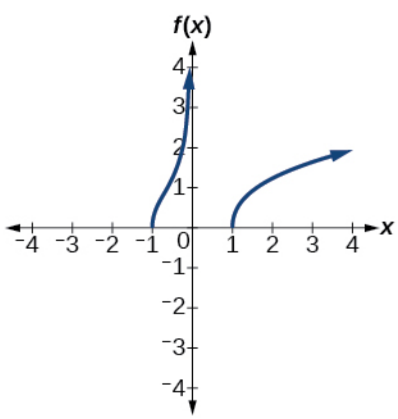
For the following exercises, use a graph to help determine the domain of the functions.

41. $f(x)=\sqrt{\frac{(x+1)(x-1)}{x}}$

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+1)(x-1)}{x}\geq 0$.The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes.. For this equation, the graph could change signs at $x=0,1,\text{and } -1.$

In TI calculator, key $Y1=\sqrt{\frac{(x+1)(x-1)}{x}}$

The output graph is shown.



From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-1 \le x < 0$ or $x \ge 1$, or in interval notation, $[-1, 0) \cup [1, \infty)$.

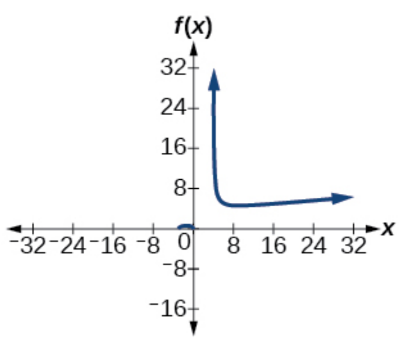
42. -

43. $f(x) = \sqrt{\dfrac{x(x+3)}{x-4}}$

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\dfrac{x(x+3)}{x-4} \ge 0$. The output of a rational function can change signs (change from positive to negative or vice versa) at x-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = 4, 0, \text{and} -3.$

In TI calculator, key $Y1 = \sqrt{\dfrac{x(x+3)}{x-4}}$

The output graph is shown.



From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-3 \le x \le 0$ or $x > 4$, or in interval notation, $[-3, 0] \cup (4, \infty)$.

44. -

45. $f(x) = \sqrt{\dfrac{9-x^2}{x+4}}$

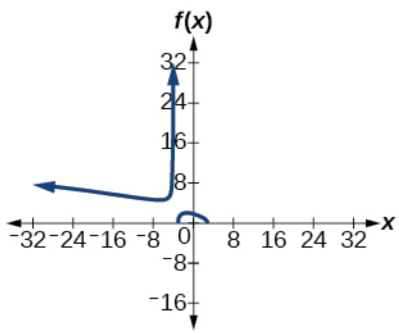
Factor the numerator and the denominator.

$$f(x)=\sqrt{\frac{9-x^2}{x+4}}$$
$$=\sqrt{\frac{(3+x)(3-x)}{x+4}}$$

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(3+x)(3-x)}{x+4}\geq 0.$ The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x=-3,3,\text{and}\,4.$

In TI calculator, key $Y1=\sqrt{\frac{9-x^2}{x+4}}$

The output graph is shown.



From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-\infty\leq x<-4$ or $-3\leq x\leq 3,$ or in interval notation, $[-\infty,-4)\cup[-3,3].$

Technology

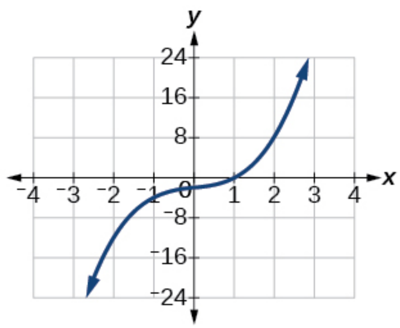
For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with *y*-coordinates given.

46. -

47. $f(x)=x^3+x-2,\ y=0,1,2$

In TI calculator, key $Y1=x^3+x-2$

The output graph is shown.



If (a,b) is on the graph of f, then (b,a) is on the graph of f^{-1}.

Since $(0,-2)$ is on the graph of f, then $(-2,0)$ is on the graph of f^{-1}.

Similarly, since $(1,0)$ is on the graph of f, then $(0,1)$ is on the graph of f^{-1}.

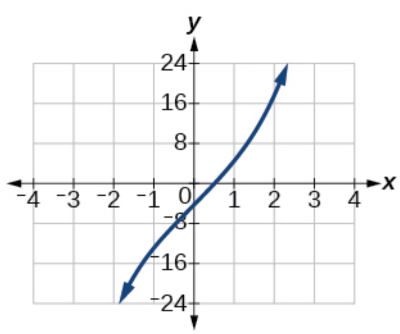
Similarly, since $(2,8)$ is on the graph of f, then $(8,2)$ is on the graph of f^{-1}.

48. -

49. $f(x)=x^3+8x-4, y=-1,0,1$

In TI calculator, key $Y1=x^3+8x-4$

The output graph is shown.



If (a,b) is on the graph of f, then (b,a) is on the graph of f^{-1}.

Since $(-1,-13)$ is on the graph of f, then $(-13,-1)$ is on the graph of f^{-1}.

Similarly, since $(0,-4)$ is on the graph of f, then $(-4,0)$ is on the graph of f^{-1}.

Similarly, since $(1,5)$ is on the graph of f, then $(5,1)$ is on the graph of f^{-1}.

50. -

Extensions

For the following exercises, find the inverse of the functions with *a*, *b*, and *c* positive real numbers.

51. $f(x) = ax^3 + b$

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = ax^3 + b$ Interchange x and y.

$x = ay^3 + b$ Subtract b from both sides.

$x - b = ay^3$ Divide both sides by a.

$\dfrac{x-b}{a} = y^3$ Take cubic root on both sides.

$\sqrt[3]{\dfrac{x-b}{a}} = y$ Rename the function $f^{-1}(x)$.

$$f^{-1}(x) = \sqrt[3]{\frac{x-b}{a}}$$

52. -

53. $f(x) = \sqrt{ax^2 + b}$

The original function $f(x) = \sqrt{ax^2 + b}$ is not one-to-one, but the function is restricted to a domain of $x \ge 0$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$y=\sqrt{ax^2+b}$$ Interchange x and y.

$$x=\sqrt{ay^2+b}$$ Square on each side.

$$x^2=ay^2+b$$ Subtract b from both sides.

$$x^2-b=ay^2$$ Divide both sides by a.

$$\frac{x^2-b}{a}=y^2$$ Take the square root.

$$\pm\sqrt{\frac{x^2-b}{a}}=y$$

The domain of the original function was restricted to $x\geq 0,$ so the outputs of the inverse need to be the same, $f^{-1}\left(x\right)\geq 0,$ and we must use the + case:

$$f^{-1}(x)=\sqrt{\frac{x^2-b}{a}}$$

54. -

55. $f(x)=\dfrac{ax+b}{x+c}$

This is a transformation of the basic rational function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$y=\frac{ax+b}{x+c}$$ Interchange x and y.

$$x=\frac{ay+b}{y+c}$$ Multiply $y+c$ on both sides.

$$x\left(y+c\right)=ay+b$$ Distribute x.

$$xy+cx=ay+b$$ Subtract xy and b from both sides.

$$cx-b=ay-xy$$ Divide $a-x$ by both sides.

$$\frac{cx-b}{a-x}=y$$ Rename the function $f^{-1}\left(x\right)$.

$$f^{-1}\left(x\right)=\frac{cx-b}{a-x}$$

Real-World Applications

For the following exercises, determine the function described and then use it to answer the question.

56. -

57. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.

Express t as a function of height, h, by moving the terms involving on the time, t to one side and the remaining terms on the other side.

$h = 600 - 16t^2$	Subtract h from both sides.
$0 = 600 - h - 16t^2$	Add $16t^2$ to both sides.
$16t^2 = 600 - h$	Divide both sides by 16.
$t^2 = \dfrac{600 - h}{16}$	Take the square root.
$t = \sqrt{\dfrac{600 - h}{16}}$	Rename the function $t(h)$.
$t(h) = \sqrt{\dfrac{600 - h}{16}}$	

Note that we use the positive square root because of the context of the problem. Time is positive. Substitute $h = 400$ in the expression, $t(h) = \sqrt{\dfrac{600 - h}{16}}$.

$$t(400) = \sqrt{\frac{600 - 400}{16}} = \sqrt{\frac{200}{16}} \approx 3.54$$

The object required 3.54 seconds to reach a height of 400 feet.

58. -

59. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.

Express r as a function of area, A, by moving the terms involving on the time, r to one side and the remaining terms on the other side.

$$A = 4\pi r^2 \qquad \text{Divide both sides by } 4\pi.$$

$$\frac{A}{4\pi} = r^2 \qquad \text{Take the square root.}$$

$$\sqrt{\frac{A}{4\pi}} = r \qquad \text{Rename the function } r(A).$$

$$r(A) = \sqrt{\frac{A}{4\pi}}$$

Note that we use the positive square root because of the context of the problem. Radius is positive. Substitute $A = 1000$ in the expression, $r(A) = \sqrt{\frac{A}{4\pi}}$.

$$r(A) = \sqrt{\frac{A}{4\pi}} = \sqrt{\frac{1000}{4\pi}} \approx 8.92$$

The radius of the given sphere is 8.92 inches.

60. -

61. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.

Solve for l in terms of T.

$$T = 2\pi\sqrt{\frac{l}{32.2}} \qquad \text{Divide both sides by } 2\pi.$$

$$\frac{T}{2\pi} = \sqrt{\frac{l}{32.2}} \qquad \text{Square each side.}$$

$$\left(\frac{T}{2\pi}\right)^2 = \frac{l}{32.2} \qquad \text{Multiply both sides by 32.2.}$$

$$32.2\left(\frac{T}{2\pi}\right)^2 = l \qquad \text{Rename the function as, } l(T).$$

$$l(T) = 32.2\left(\frac{T}{2\pi}\right)^2$$

Substitute $T = 2$ in the expression, $l(T) = 32.2\left(\frac{T}{2\pi}\right)^2$.

$$l(2) = 32.2\left(\frac{2}{2\pi}\right)^2 = 32.2\left(\frac{1}{\pi}\right)^2 \approx 3.26$$

The length of the given pendulum is 3.26 feet.

62. -

63. The surface area, $A,$ of a cylinder in terms of its radius, $r,$ and height, $h,$ is given by $A = 2\pi r^2 + 2\pi rh.$ If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.

Solve for r in terms of A by substituting the value of h.

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh && \text{Substitute } h = 4. \\ A &= 2\pi r^2 + 2\pi r(4) && \text{Simplify.} \\ A &= 2\pi r^2 + 8\pi r && \text{Subtract } A \text{ from both sides.} \\ 0 &= 2\pi r^2 + 8\pi r - A \end{aligned}$$

Use the quadratic formula to solve for r. We have $a = 2\pi, b = 8\pi,$ and $c = -A.$

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-8\pi \pm \sqrt{(8\pi)^2 - 4(2\pi)(-A)}}{2(2\pi)} \\ &= \frac{-8\pi \pm \sqrt{64\pi^2 + 8\pi A}}{4\pi} \\ &= \frac{-8\pi}{4\pi} \pm \frac{2\sqrt{2\pi}\sqrt{8\pi + A}}{4\pi} \\ &= -2 \pm \frac{\sqrt{8\pi + A}}{\sqrt{2\pi}} \end{aligned}$$

Only positive numbers make sense as dimensions for a cylinder, so we need not test any negative values.

$$r(A) = \sqrt{\frac{A + 8\pi}{2\pi}} - 2$$

Substitute $A = 200$ in the expression, $r(A) = \sqrt{\frac{A + 8\pi}{2\pi}} - 2.$

$$r(200) = \sqrt{\frac{200 + 8\pi}{2\pi}} - 2$$

The radius of the given cylinder is 3.99 feet.

64. -

65. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

Solve for r in terms of V by substituting the value of h.

$V = \frac{1}{3}\pi r^2 h$	Substitute $h = 30$.
$V = \frac{1}{3}\pi r^2 (30)$	Simplify.
$V = 10\pi r^2$	Divide both sides by 10π.
$\frac{V}{10\pi} = r^2$	Take the square root.
$\sqrt{\frac{V}{10\pi}} = r$	Rename the function, as $r(V)$.
$\sqrt{\frac{V}{10\pi}} = r(V)$	

Note that we use the positive square root because of the context of the problem. Radius is positive. Substitute $V = 1000$ in the expression, $r(V) = \sqrt{\frac{V}{10\pi}}$.

$$r(1000) = \sqrt{\frac{1000}{10\pi}} = \sqrt{\frac{100}{\pi}} \approx 5.64$$

The radius of the given cone is 5.64 feet.

Chapter 5
Polynomial and Rational Functions
5.8 Modeling Using Variation

Verbal

1. What is true of the appearance of graphs that reflect a direct variation between two variables?
   The graph will have the appearance of a power function.
2. -
   One variable will equal a constant divided by the other variable raised to a power.
3. Is there a limit to the number of variables that can jointly vary? Explain.
   No. Multiple variables may jointly vary.

Algebraic

For the following exercises, write an equation describing the relationship of the given variables.

4. -

5. y varies directly as the square of x and when $x = 4,\ y = 80$.

   The general formula for direct variation with a square is $y = kx^2$. The constant can be found by dividing y by the square of x.

   $$\begin{aligned} k &= \frac{y}{x^2} \\ &= \frac{80}{4^2} \\ &= 5 \end{aligned}$$

   Now use the constant to write an equation that represents this relationship.

   $y = 5x^2$

6. -

7. y varies directly as the cube of x and when $x = 2, y = 80$.

   The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$k = \frac{y}{x^3}$$
$$= \frac{80}{2^3}$$
$$= \frac{80}{8}$$
$$= 10$$

Now use the constant to write an equation that represents this relationship.

$$y = 10x^3$$

8. -
9. y varies directly as the fourth power of x and when $x = 1$, $y = 6$.

The general formula for direct variation with the fourth power is $y = kx^4$. The constant can be found by dividing y by the fourth power of x.

$$k = \frac{y}{x^4}$$
$$= \frac{6}{1^4}$$
$$= \frac{6}{1}$$
$$= 6$$

Now use the constant to write an equation that represents this relationship.

$$y = 6x^4$$

10. -

11. y varies inversely as the square of x and when $x = 3$, $y = 2$.

The general formula for inverse variation with the square is $y = \frac{k}{x^2}$. The constant can be found by multiplying y by the square of x.

$$k = x^2 y = 3^2 \cdot 2 = 18$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^2}, k = 18$$
$$y = \frac{18}{x^2}$$

12. -

13. y varies inversely as the fourth power of x and when $x = 3,\ y = 1.$

The general formula for inverse variation with the fourth power is $y = \frac{k}{x^4}$. The constant can be found by multiplying y by the fourth power of $x.$

$$k = x^4 y = 3^4 \cdot 1 = 81$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^4},\ k = 81$$

$$y = \frac{81}{x^4}$$

14. -

15. y varies inversely as the cube root of x and when $x = 64,\ y = 5.$

The general formula for inverse variation with the cube root is $y = \frac{k}{\sqrt[3]{x}}$. The constant can be found by multiplying y by the cube root of $x.$

$$k = \sqrt[3]{x} \cdot y = \sqrt[3]{64} \cdot 5 = 20$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{\sqrt[3]{x}},\ k = 20$$

$$y = \frac{20}{\sqrt[3]{x}}$$

16. -

17. y varies jointly as $x, z,$ and w and when $x = 1,\ z = 2,\ w = 5,$ then $y = 100.$

Begin by writing an equation to show the relationship between the variables.

$$y = kxzw$$

Substitute $x = 1,\ z = 2,\ w = 5,$ and $y = 100$ to find the value of the constant $k.$

$$100 = k(1)(2)(5)$$

$$\frac{100}{10} = k$$

$$10 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 10xzw$$

18. -

19. y varies jointly as x and the square root of z and when $x = 2$ and $z = 25$, then $y = 100$.

Begin by writing an equation to show the relationship between the variables.

$$y = kx\sqrt{z}$$

Substitute $x = 2$, $z = 25$, and $y = 100$ to find the value of the constant k.

$$100 = k(2)\sqrt{25}$$

$$100 = k(2)(5)$$

$$\frac{100}{10} = k$$

$$10 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 10x\sqrt{z}$$

20. -

21. y varies jointly as x and z and inversely as w. When $x = 3$, $z = 5$, and $w = 6$, then $y = 10$.

Begin by writing an equation to show the relationship between the variables.

$$y = k\frac{xz}{w}$$

Substitute $x = 3$, $z = 5$, $w = 6$ and $y = 10$ to find the value of the constant k.

$$10 = k\frac{(3)(5)}{(6)}$$

$$\frac{60}{15} = k$$

$$4 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 4\frac{xz}{w}$$

22. -

23. y varies jointly as x and z and , inversely as the square root of w and the square of t.

When $x = 3$, $z = 1$, $w = 25$, and $t = 2$, then $y = 6$.

Begin by writing an equation to show the relationship between the variables.

$$y = k\frac{xz}{\sqrt{w}t^2}$$

Substitute $x = 3,\ z = 1,\ w = 25,\ t = 2$ and $y = 6$ to find the value of the constant $k.$

$$6 = k\frac{(3)(1)}{\sqrt{25}(2)^2}$$

$$6 = k\frac{(3)}{(5)(4)}$$

$$\frac{6 \cdot 20}{3} = k$$

$$40 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 40\frac{xz}{\sqrt{w}t^2}$$

Numeric

For the following exercises, use the given information to find the unknown value.

24. -

25. y varies directly as the square of $x.$ When $x = 2,$ then $y = 16.$ Find y when $x = 8.$

The general formula for direct variation with a square is $y = kx^2.$ The constant can be found by dividing y by square of $x.$

$$k = \frac{y}{x^2} = \frac{16}{2^2} = 4$$

Now use the constant to write an equation that represents this relationship.

$$y = 4x^2$$

Substitute $x = 8$ and solve for $y.$

$$y = 4(8)^2 = 256$$

26. -

27. y varies directly as the square root of $x.$ When $x = 16,$ then $y = 4.$ Find y when $x = 36.$

The general formula for direct variation with a square root is $y = k\sqrt{x}$. The constant can be found by dividing y by square root of x.

$$k = \frac{y}{\sqrt{x}}$$
$$= \frac{4}{\sqrt{16}}$$
$$= 1$$

Now use the constant to write an equation that represents this relationship.

$$y = \sqrt{x}$$

Substitute $x = 36$ and solve for y.

$$y = \sqrt{36} = 6$$

28. -

29. y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

The general formula for inverse variation is $y = \frac{k}{x}$. The constant can be found by multiplying y by x.

$$k = xy = 3 \cdot 2 = 6$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x}, k = 6$$

$$y = \frac{6}{x}$$

Substitute $x = 1$ and solve for y.

$$y = \frac{6}{1} = 6$$

30. -

31. y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by cube of x.

$$k = x^3 y = 3^3 \cdot 1 = 27$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^3}, k = 27$$

$y=\frac{27}{x^3}$

Substitute $x=1$ and solve for y.

$y=\frac{27}{1^3}=27$

32. -

33. y varies inversely with the cube root of x. When $x=27$, then $y=5$. Find y when $x=125$.

The general formula for inverse variation with a cube root is $y=\frac{k}{\sqrt[3]{x}}$. The constant can be found by multiplying y by cube root of x.

$k=\sqrt[3]{x}\cdot y=\sqrt[3]{27}\cdot 5=3\cdot 5=15$

Now we use the constant to write an equation that represents this relationship.

$y=\frac{k}{\sqrt[3]{x}}, k=15$

$y=\frac{15}{\sqrt[3]{x}}$

Substitute $x=125$ and solve for y.

$y=\frac{15}{\sqrt[3]{125}}=\frac{15}{5}=3$

34. -

35. y varies jointly as x, z, and w. When $x=2$, $z=1$, and $w=12$, then $y=72$. Find y when $x=1$, $z=2$ and $w=3$.

Begin by writing an equation to show the relationship between the variables.

$y=kxzw$

Substitute $x=2$, $z=1$, $w=12$, and $y=72$ to find the value of the constant k.

$72=k(2)(1)(12)$

$\frac{72}{24}=k$

$3=k$

Now we can substitute the value of the constant into the equation for the relationship.

$y=3xzw$

Substitute $x=1$, $z=2$, $w=3$ and solve for y.

$y=3(1)(2)(3)=18$

36. -

37. y varies jointly as the square of x and the square root of z. When $x=2$ and $z=9$, then $y=24$. Find y when $x=3$ and $z=25$.

Begin by writing an equation to show the relationship between the variables.

$$y = kx^2\sqrt{z}$$

Substitute $x=2$, $z=9$, and $y=24$. to find the value of the constant k.

$$24 = k(2)^2\sqrt{9}$$
$$24 = k(4)(3)$$
$$\frac{24}{12} = k$$
$$2 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 2x^2\sqrt{z}$$

Substitute $x=3$, $z=25$ and solve for y.

$$y = 2(3)^2\sqrt{25} = 2(9)(5) = 90$$

38. -

39. y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x=2$, $z=2$, and $w=64$, then $y=12$. Find y when $x=1, z=3$, and $w=4$.

Begin by writing an equation to show the relationship between the variables.

$$y = \frac{kx^2z^3}{\sqrt{w}}$$

Substitute $x=2$, $z=2$, $w=64$, and $y=12$ to find the value of the constant k.

$$12 = \frac{k2^22^3}{\sqrt{64}}$$
$$12(8) = k(32)$$
$$\frac{96}{32} = k$$
$$3 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y=\frac{3x^2z^3}{\sqrt{w}}$$

Substitute $x=1, z=3, w=4$ and solve for y.

$$y=\frac{3x^2z^3}{\sqrt{w}}=\frac{3(1)^2(3)^3}{\sqrt{4}}=\frac{81}{2}$$

40. -

Technology

For the following exercises, use a calculator to graph the equation implied by the given variation.

41. y varies directly with the square of x and when $x=2,\ y=3$.

The general formula for direct variation with a square is $y=kx^2$. The constant can be found by dividing y by the square of x.

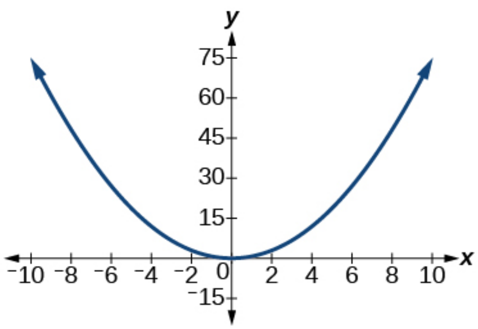
$$\begin{aligned} k &= \frac{y}{x^2} \\ &= \frac{3}{2^2} \\ &= \frac{3}{4} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y=\frac{3}{4}x^2$$

In TI calculator, key $Y1=\frac{3}{4}x^2$

The resulting graph is as shown.



42. -

43. y varies directly as the square root of x and when $x = 36,\ y = 2.$

The general formula for direct variation with a square root is $y = k\sqrt{x}.$ The constant can be found by dividing y by the square root of $x.$

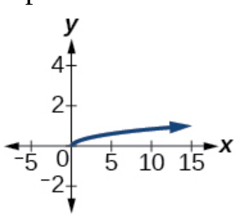
$$\begin{aligned} k &= \frac{y}{\sqrt{x}} \\ &= \frac{2}{\sqrt{36}} \\ &= \frac{1}{3} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{1}{3}\sqrt{x}$$

In TI calculator, key $Y1 = \frac{1}{3}\sqrt{x}$

The resulting graph is as shown.



44. -

45. y varies inversely as the square of x and when $x = 1,\ y = 4.$

The general formula for inverse variation with the square is $y = \frac{k}{x^2}$. The constant can be found by multiplying y by the square of $x.$

$$k = x^2 y = 1^2 \cdot 4 = 4$$

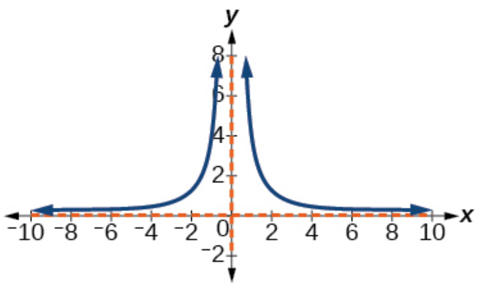
Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^2},\ k = 4$$

$$y = \frac{4}{x^2}$$

In TI calculator, key $Y1 = \frac{4}{x^2}$

The resulting graph is as shown.



Extensions

For the following exercises, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

46. -

47. Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

Substitute $a = 142$ in $T^2 = \frac{1}{804357}a^3$ to find the value of T.

$$T^2 = \frac{1}{804357}(142)^3$$

$$T = \sqrt{\frac{1}{804357}(142)^3} \qquad \text{Take the square root.}$$

$$\approx 1.89$$

The time required for Mars to orbit the Sun is 1.89 years.

48. -

49. Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

Substitute $a = 108$ in $T^2 = \frac{1}{3375000}a^3$ to find the value of T.

$$T^2 = \frac{1}{3375000}(108)^3$$

$$T = \sqrt{\frac{1}{3375000}(108)^3} \quad \text{Take the square root.}$$

$$\approx 0.61$$

The time required for Venus to orbit the Sun is 0.61 years.

50. -

Real-World Applications

For the following exercises, use the given information to answer the questions.

51. The distance, s, that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

Begin by writing an equation to show the relationship between the variables.

$$s = kt^2$$

Substitute $s = 16$ and $t = 1$ to find the value of the constant k.

$$16 = k(1)^2$$

$$16 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$s = 16t^2$$

Substitute $s = 144$ and solve for t.

$$144 = 16t^2$$

$$\frac{144}{16} = t^2$$

$$9 = t^2$$

$$\pm 3 = t$$

Only positive numbers make sense for the time taken.

The object takes 3 seconds.

52. -

53. The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

Let s be the rate of vibration of a string and l be the length of the string.

Begin by writing an equation to show the relationship between the variables.

$$s = \frac{k}{l}$$

Substitute $s = 128$ and $l = 24$ to find the value of the constant k.

$$128 = \frac{k}{24}$$

$$128 \cdot 24 = k$$

$$3072 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$s = \frac{3072}{l}$$

Substitute $s = 64$ and solve for l.

$$64 = \frac{3072}{l}$$

$$l = \frac{3072}{64}$$

$$= 48$$

The length of the string is 48 inches.

54. -

55. The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

Let *w* be the weight of an object above the surface of the Earth and *d* be the distance from the center of the Earth.

Begin by writing an equation to show the relationship between the variables.

$$w = \frac{k}{d^2}$$

Substitute $w = 50$ and $d = 3960$ to find the value of the constant k.

$$50 = \frac{k}{3960^2}$$

$$784080000 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$w = \frac{784080000}{d^2}$$

Substitute $d = 3970$ and solve for w.

$$w = \frac{784080000}{3970^2} \approx 49.75$$

The body weighs 49.75 pounds.

56. -

57. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

Let C be the current in a circuit and R be the resistance.

Begin by writing an equation to show the relationship between the variables.

$$C = \frac{k}{R}$$

Substitute $C = 40$ and $R = 10$ to find the value of the constant k.

$$40 = \frac{k}{10}$$

$$400 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$C = \frac{400}{R}$$

Substitute $R = 12$ and solve for C.

$$C = \frac{400}{12} \approx 33.33$$

The current in the circuit is 33.33 amperes.

58. -

59. The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

Let H be the horsepower that a shaft can safely transmit, s be the speed and d be the diameter.

Begin by writing an equation to show the relationship between the variables.

$$H = ksd^3$$

Substitute $H = 45, s = 100$ and $d = 3$ to find the value of the constant k.

$$45 = k(100)(3)^3$$

$$\frac{45}{2700} = k$$

$$\frac{1}{60} = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$H = \frac{1}{60}sd^3$$

Substitute $H = 60$ and $s = 150$ and solve for d.

$$60 = \frac{1}{60}(150)d^3$$

$$\frac{3600}{150} = d^3$$

$$24 = d^3$$

$$d \approx 2.88$$

The diameter is 2.88 inches.

60. -

Chapter 5 Review Exercises

Section 5.1

For the following exercises, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.

1. $f(x)=x^2-4x-5$

   Because a is positive, the parabola opens upward and has a minimum value. We need to determine the minimum value. We can begin by finding the *x*-value of the vertex.

   $$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-4}{2\cdot 1} \\ &= 2 \end{aligned}$$

   The minimum value is given by $f(h)$.

   $$\begin{aligned} f(2) &= (2)^2-4(2)-5 \\ &= 4-8-5 \\ &= -9 \end{aligned}$$

   The vertex is at $(2,-9)$.

   Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

   $$f(x)=ax^2+bx+c$$

   $$f(x)=x^2-4x-5$$

   Using the vertex to determine the shifts, $f(x)=(x-2)^2-9$

   Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x=2$.

   We find the *y*- intercept by evaluating $f(0)$.

   $$\begin{aligned} f(0) &= (0)^2-4(0)-5 \\ &= -5 \end{aligned}$$

   So the *y*- intercept is at $(0,\ -5)$.

   For the *x*-intercepts, we find all solutions of $f(x)=0$.

   $$x^2-4x-5=0$$

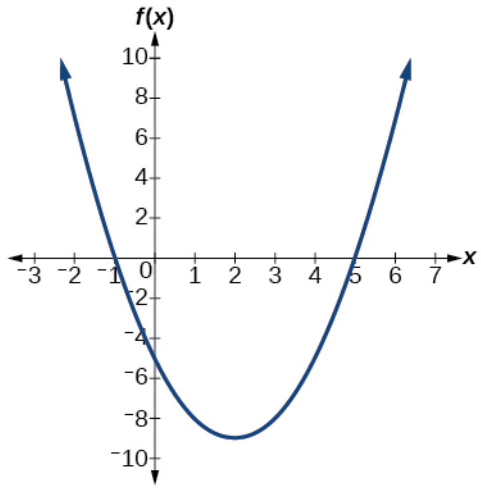
   In this case, the quadratic can be factored easily, providing the simplest method for solution.

   $$(x+1)(x-5)=0$$

$$\begin{aligned} x+1&=0 \quad \text{or} \quad x-5=0 \\ x&=-1 \qquad\qquad x=5 \end{aligned}$$

So the *x*-intercepts are at $(-1,\ 0)$ and $(5,\ 0)$.

The graph for the given function is as shown.



2. -

For the following exercises, find the equation of the quadratic function using the given information.

3. The vertex is $(-2, 3)$ and a point on the graph is $(3, 6)$.

   Substitute the values of h and k in the function

$$\begin{aligned} f(x) &= a(x-h)^2+k \\ &= a(x-(-2))^2+3 \\ &= a(x+2)^2+3 \end{aligned}$$

   Substituting the coordinates of a point on the curve, such as $(3,\ 6)$, we can solve for the stretch factor.

$$\begin{aligned} 6 &= a(3+2)^2+3 \\ 6 &= 25a+3 \\ 3 &= 25a \\ \frac{3}{25} &= a \end{aligned}$$

   In standard form, the algebraic model for this graph is $f(x)=\frac{3}{25}(x+2)^2+3.$

4. -

For the following exercises, compete the task.

5. A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

   Let x be the length of the shorter side $600-2x$ be the length of the longer side.

   Then their product will be $f(x)=x(600-2x)=-2x^2+600x$.

   The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value.

   We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{600}{2\cdot(-2)} \\ &= 150 \end{aligned}$$

   The maximum value is given by $f(h)$.

$$\begin{aligned} f(150) &= -2(150)^2+600(150) \\ &= 45000 \end{aligned}$$

   Thus, the dimension of the plot is $600-2(150)=300$ meters by 150 meters.

6. -

Section 5.2

For the following exercises, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

7. $f(x)=4x^5-3x^3+2x-1$

   Yes; $f(x)$ is a polynomial function because it consists of the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

   The highest power of x is 5, so the degree is 5. The leading term is the term containing that degree, $4x^5$. The leading coefficient is the coefficient of that term, 4.

8. -

9. $f(x)=x^2\left(3-6x+x^2\right)$

   Obtain the general form by expanding the given expression.

$$\begin{aligned} f(x) &= x^2\left(3-6x+x^2\right) \\ &= 3x^2-6x^3+x^4 \end{aligned}$$

Yes; $f(x)$ is a polynomial function because it consists of the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.
The highest power of x is 4, so the degree is 4. The leading term is the term containing that degree, x^4. The leading coefficient is the coefficient of that term, 1.

For the following exercises, determine end behavior of the polynomial function.

10. -
11. $f(x) = 4x^3 - 6x^2 + 2$

    The leading term is $4x^3$; therefore, the degree of the polynomial is 3. The degree is odd (3) and the leading coefficient is positive (4), so the end behavior is
    as $x \to -\infty, f(x) \to -\infty$
    as $x \to \infty, f(x) \to \infty.$
12. -

Section 5.3

For the following exercises, find all zeros of the polynomial function, noting multiplicities.

13. $f(x) = (x+3)^2(2x-1)(x+1)^3$

    The zero associated with the factor, $x = -3,$ has multiplicity 2 because the factor $(x+3)$ occurs twice. The next zero associated with the factor, $x = \frac{1}{2},$ has multiplicity 1 because the factor $(2x-1)$ is linear (has a degree of 1). The last zero associated with the factor, $x = -1,$ has multiplicity 3 because the factor $(x+1)$ is cubic (degree 3).
14. --

15. $f(x) = x^3 - 4x^2 + x - 4$

    Find zeros for $f(x)$ by factoring.

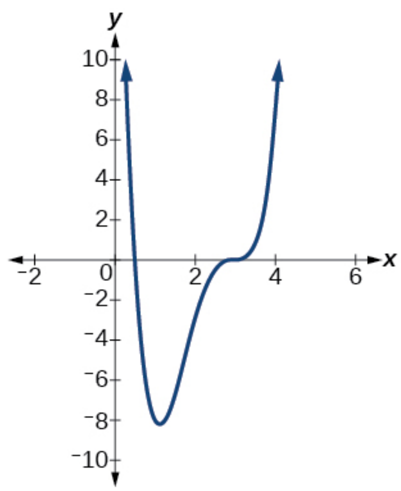
    $$\begin{aligned} f(x) &= x^3 - 4x^2 + x - 4 \\ &= (x^2+1)(x-4) \end{aligned}$$

    The zero associated with the factor, $x = 4,$ has multiplicity 1 because the factor $(x-4)$ is linear (has a degree of 1).

For the following exercises, based on the given graph, determine the zeros of the function and note multiplicity.

16. -

17.



Starting from the left, the first zero occurs at $x = \frac{1}{2}$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The next zero occurs at $x = 3$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3.

18. -

Section 5.4

For the following exercises, use long division to find the quotient and remainder.

19. $\dfrac{x^3 - 2x^2 + 4x + 4}{x - 2}$

$$\begin{array}{rll} & x^2 + 4 & \\ x-2 \enclose{longdiv}{} & x^3 - 2x^2 + 4x + 4 & x^3 \text{ divided by } x \text{ is } x^2. \\ & \underline{-(x^3 - 2x^2)} & \text{Multiply } x-2 \text{ by } x^2. \\ & 4x + 4 & \text{Subtract. Bring down the next 2 terms. } 4x \text{ divided by } x \text{ is } 4. \\ & \underline{-(4x - 8)} & \text{Multiply } x-2 \text{ by } 4. \\ & 12 & \text{Subtract.} \end{array}$$

The quotient is $x^2 + 4$. The remainder is 12.

20. -

For the following exercises use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

21. $\dfrac{x^3-2x^2+5x-1}{x+3}$

The binomial divisor is $x+3$ so $k=-3$.

Add each column, multiply the result by –3, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -3 & 1 & -2 & 5 & -1 \\ & & -3 & 15 & -60 \\ \hline & 1 & -5 & 20 & -61 \end{array}$$

The quotient is $x^2-5x+20$. The remainder is -61.

Thus, $\dfrac{x^3-2x^2+5x-1}{x+3}=x^2-5x+20-\dfrac{61}{x+3}$

22. -

23. $\dfrac{2x^3+6x^2-11x-12}{x+4}$

The binomial divisor is $x+4$ so $k=-4$.

Add each column, multiply the result by –4, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -4 & 2 & 6 & -11 & -12 \\ & & -8 & 8 & 12 \\ \hline & 2 & -2 & -3 & 0 \end{array}$$

The quotient is $2x^2-2x-3$. The remainder is 0 .

Thus, $\dfrac{2x^3+6x^2-11x-12}{x+4}=2x^2-2x-3$

Write the result in factored form.

$2x^3+6x^2-11x-12=\left(2x^2-2x-3\right)\left(x+4\right)$

24. -

Section 5.5

For the following exercises, use the Rational Zero Theorem to solve the polynomial equation.

25. $2x^3-3x^2-18x-8=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –8 and q is a factor of 2.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of }-8}{\text{factor of 2}}$$

The factors of 2 are ± 1 and ± 2 and the factors of –8 are $\pm1,\pm2,\pm4$ and $\pm 8.$ The possible values for $\frac{p}{q}$ are $\pm\frac{1}{2},\pm1,\pm2,\pm4$ and $\pm8.$

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with -2.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & -18 & -8 \\ & & -4 & 14 & 8 \\ \hline & 2 & -7 & -4 & 0 \end{array}$$

Dividing by $(x+2)$ gives a remainder of 0, so -2 is a zero of the function. The polynomial can be written as

$(x+2)(2x^2-7x-4).$

We can factor the quadratic factor to write the polynomial as

$(2x^2-7x-4)=(2x+1)(x-4)$

To find the other zeros, set each factor equal to 0.

$$(2x+1)=0 \quad \text{or} \quad (x-4)=0$$
$$x=-\frac{1}{2} \qquad\qquad x=4$$

The solutions of $2x^3-3x^2-18x-8=0$ are $-2,4,-\frac{1}{2}.$

26. -

27. $2x^4-17x^3+46x^2-43x+12=0$

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 12 and q is a factor of 2.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of 12}}{\text{factor of 2}}$$

The factors of 2 are ±1 and ±2 and the factors of 12 are $\pm1, \pm2, \pm3, \pm4, \pm6$ and ±12. The possible values for $\frac{p}{q}$ are $\pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm4, \pm6$ and ±12.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

$$\begin{array}{r|rrrrr} 1 & 2 & -17 & 46 & -43 & 12 \\ & & 2 & -15 & 31 & -12 \\ \hline & 2 & -15 & 31 & -12 & 0 \end{array}$$

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(2x^3-15x^2+31x-12)$$

Again, use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's consider 3.

$$\begin{array}{r|rrrr} 3 & 2 & -15 & 31 & -12 \\ & & 6 & -27 & 12 \\ \hline & 2 & -9 & 4 & 0 \end{array}$$

Dividing by $(x-3)$ gives a remainder of 0, so 3 is a zero of the function. The polynomial can be written as

$$(x-1)(x-3)(2x^2-9x+4).$$

We can factor the quadratic factor to write the polynomial as

$$(2x^2-9x+4) = (2x-1)(x-4)$$

To find the other zeros, set each factor equal to 0.

$$(2x-1)=0 \quad \text{or} \quad (x-4)=0$$

$$x=\frac{1}{2} \qquad\qquad x=4$$

The solutions of $2x^4-17x^3+46x^2-43x+12=0$ are $1, 3, 4, \frac{1}{2}$.

28. -

For the following exercises, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.

29. $x^3-3x^2-2x+4=0$

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = (-x)^3 - 3(-x)^2 - 2(-x) + 4$$
$$= -x^3 - 3x^2 + 2x + 4$$

There is only one sign change, so there is 1 negative real root.

30. -

Section 5.6

For the following rational functions, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph.

31. $f(x) = \dfrac{x+2}{x-5}$

We can find the y-intercept by evaluating the function at zero.

$$f(0) = \frac{0+2}{0-5} = -\frac{2}{5}$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{x+2}{x-5}$$ This is zero when the numerator is zero.

$$0 = x + 2$$
$$-2 = x$$

The y-intercept is $\left(0, -\frac{2}{5}\right)$, the x-intercept is $(-2, 0)$.

At the x-intercept, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

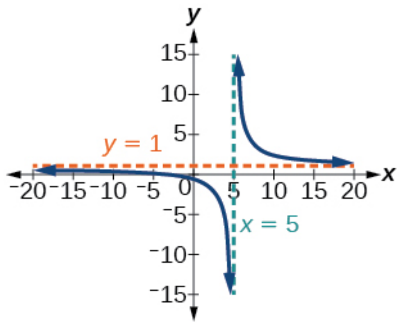
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x - 5 = 0$$
$$x = 5$$

The vertical asymptote is $x = 5$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{1}{1}$ or $y = 1$.

The graph of the given function is shown.



32. -

33. $f(x)=\dfrac{3x^2-27}{x^2+x-2}$

First, factor the numerator and denominator.

$$f(x)=\frac{3x^2-27}{x^2+x-2}$$
$$=\frac{3(x+3)(x-3)}{(x-1)(x+2)}$$

We can find the *y*-intercept by evaluating the function at zero.

$$f(0)=\frac{3(0+3)(0-3)}{(0-1)(0+2)}=\frac{27}{2}$$

The *x*-intercepts will occur when the function is equal to zero:

$$0=\frac{3(x+3)(x-3)}{(x-1)(x+2)}$$ This is zero when the numerator is zero.

$$0=3(x+3)(x-3)$$

Set each factor equal to zero.

$x+3=0$ or $x-3=0$

$x=-3$ $\quad$ $x=3$

The *y*-intercept is $\left(0,\dfrac{27}{2}\right)$, the *x*-intercepts are $(-3,0),(3,0)$.

At the *x*-intercept, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

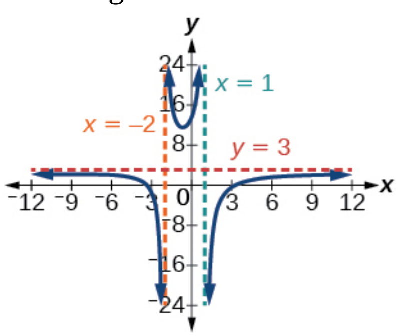
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x+2=0 \qquad \text{or} \qquad x-1=0$$
$$x=-2 \qquad\qquad\qquad x=1$$

The vertical asymptotes are $x=-2$ and $x=1$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y=\frac{3}{1}$ or $y=3$.

The graph of the given function is shown.



34. -

For the following exercises, find the slant asymptote.

35. $f(x)=\dfrac{x^2-1}{x+2}$

Since the degree of the numerator = degree of the denominator plus 1, there is a slant asymptote.

Now, the binomial divisor is $x+2$ so $k=-2$.

Add each column, multiply the result by -2, and repeat until the last column is reached.

$$\begin{array}{r|rrr} -2 & 1 & 0 & -1 \\ & & -2 & 4 \\ \hline & 1 & -2 & 3 \end{array}$$

The quotient is $x-2$ and the remainder is 3. There is a slant asymptote at $y=x-2$.

36. -

Section 5.7

For the following exercises, find the inverse of the function with the domain given.

37. $f(x)=(x-2)^2,\ x\ge 2$

The original function $f(x)=(x-2)^2$ is not one-to-one, but the function is restricted to a domain of $x\ge 2$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=(x-2)^2$ Interchange x and y.

$x=(y-2)^2$ Take the square root.

$\pm\sqrt{x}=y-2$ Add 2 to both sides.

$\pm\sqrt{x}+2=y$

The domain of the original function was restricted to $x\ge 2,$ so the outputs of the inverse need to be the same, $f^{-1}(x)\ge 2,$ and we must use the + case:

$f^{-1}(x)=\sqrt{x}+2$

38. -

39. $f(x)=x^2+6x-2,\ x\ge -3$

The original function $f(x)=x^2+6x-2$ is not one-to-one, but the function is restricted to a domain of $x\ge -3$ on which it is one-to-one.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y=x^2+6x-2$ Interchange x and y.

$x=y^2+6y-2$ Subtract x from both sides.

$0=y^2+6y-2-x$

Use the quadratic formula to solve for y. We have $a=1, b=6, \text{and } c=-2-x.$

$$
\begin{aligned}
y&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&=\frac{-6\pm\sqrt{6^2-4\cdot 1\cdot(-2-x)}}{2\cdot 1}\\
&=\frac{-6\pm\sqrt{36+4x+8}}{2}\\
&=\frac{-6\pm 2\sqrt{x+11}}{2}\\
&=-3\pm\sqrt{x+11}
\end{aligned}
$$

The domain of the original function was restricted to $x \geq -3$, so the outputs of the inverse need to be the same, $f^{-1}(x) \geq -3$, and we must use the + case:

$f^{-1}(x) = \sqrt{x+11} - 3$

40. -

41. $f(x) = \sqrt{4x+5} - 3$

The original function $f(x) = \sqrt{4x+5} - 3$ is one-to-one on its domain of $x \geq -\frac{5}{4}$ (the radicand must be non-negative.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = \sqrt{4x+5} - 3$ Interchange x and y.

$x = \sqrt{4y+5} - 3$ Add 3 to both sides.

$x + 3 = \sqrt{4y+5}$ Square each side.

$(x+3)^2 = 4y + 5$ Subtract 5 from both sides.

$(x+3)^2 - 5 = 4y$ Divide both sides by 4.

$\frac{(x+3)^2 - 5}{4} = y$ Rename the function $f^{-1}(x)$.

$$f^{-1}(x) = \frac{(x+3)^2 - 5}{4}$$

Note that the original function has range $f(x) \geq -3$.

domain of f^{-1} = range of $f = [-3, \infty)$.

42. -

Section 5.8

For the following exercises, find the unknown value.

43. y varies directly as the square of x. If when $x = 3$, $y = 36$, find y if $x = 4$.

The general formula for direct variation with a square is $y = kx^2$. The constant can be found by dividing y by square of x.

$$k = \frac{y}{x^2}$$
$$= \frac{36}{3^2}$$
$$= 4$$

Now use the constant to write an equation that represents this relationship.

$$y = 4x^2$$

Substitute $x = 4$ and solve for y.

$$y = 4(4)^2 = 64$$

44. -

45. y varies jointly as the cube of x and as z. If when $x = 1$ and $z = 2$, $y = 6$, find y if $x = 2$ and $z = 3$.

Begin by writing an equation to show the relationship between the variables.

$$y = kx^3z$$

Substitute $x = 1$, $z = 2$ and $y = 6$ to find the value of the constant k.

$$6 = k(1)^3(2)$$
$$\frac{6}{2} = k$$
$$3 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 3x^3z$$

Substitute $x = 2$, $z = 3$ and solve for y.

$$y = 3(2)^3(3) = 3(8)(3) = 72$$

46. -

For the following exercises, solve the application problem.

47. The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If a person weighs 150 pounds when he is on the surface of Earth (3,960 miles from center), find the weight of the person if he is 20 miles above the surface.

Let *w* be the weight of an object above the surface of Earth and *d* be the distance from the center of Earth.

Begin by writing an equation to show the relationship between the variables.

$$w = \frac{k}{d^2}$$

Substitute $w = 50$ and $d = 3960$ to find the value of the constant k.

$$150 = \frac{k}{3960^2}$$

$$2352240000 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$w = \frac{2352240000}{d^2}$$

Substitute $d = 3980$ and solve for w.

$$w = \frac{2352240000}{3980^2} \approx 148.5$$

The weight of the person is 148.5 pounds.

48. -

Chapter 5 Practice Test

Give the degree and leading coefficient of the following polynomial function.

1. $f(x)=x^3\left(3-6x-2x^2\right)$

   Obtain the general form by expanding the given expression.

   $$f(x)=x^3\left(3-6x-2x^2\right)$$
   $$=3x^3-6x^4-2x^5$$

   The highest power of x is 5, so the degree is 5. The leading term is the term containing that degree, $-2x^5$. The leading coefficient is the coefficient of that term, -2.

Determine the end behavior of the polynomial function.

2. -
3. $f(x)=-2x^2(4-3x-5x^2)$

   Obtain the general form by expanding the given expression.

   $$f(x)=-2x^2(4-3x-5x^2)$$
   $$=-8x^2+6x^3+10x^4$$

   The leading term is $10x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is positive (10), so the end behavior is

   $\text{as } x\to-\infty, f(x)\to\infty$

   $\text{as } x\to\infty, f(x)\to\infty.$

Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

4. -

Given information about the graph of a quadratic function, find its equation.

5. Vertex $(2,\ 0)$ and point on graph $(4,\ 12)$.

   Substitute the values of h and k in the function

   $$f(x)=a(x-h)^2+k$$
   $$=a(x-2)^2+0$$
   $$=a(x-2)^2$$

   Substituting the coordinates of a point on the curve, such as $(4,\ 12)$, we can solve for the stretch factor.

$$12 = a(4-2)^2$$
$$12 = 4a$$
$$\frac{12}{4} = a$$
$$3 = a$$

In standard form, the algebraic model for this graph is $f(x) = 3(x-2)^2$.

Solve the following application problem.

6. -

Find all zeros of the following polynomial functions, noting multiplicities.

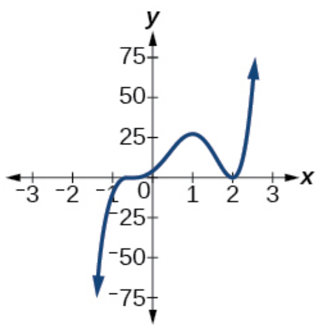
7. $f(x) = (x-3)^3(3x-1)(x-1)^2$

   The zero associated with the factor, $x = 3,$ has multiplicity 3 because the factor $(x+3)$ is cubic (degree 3). The next zero associated with the factor, $x = \frac{1}{3},$ has multiplicity 1 because the factor $(3x-1)$ is linear (has a degree of 1). The last zero associated with the factor, $x = 1,$ has multiplicity 2 because the factor $(x-1)$ occurs twice.

8. -

Based on the graph, determine the zeros of the function and multiplicities.

9.



Starting from the left, the first zero occurs at $x = -\frac{1}{2}$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. Because of its rapid turning, we know that the multiplicity is likely 3.

The next zero occurs at $x = 2$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of 2 has multiplicity 2.

Use long division to find the quotient.

10. -

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

11. $\dfrac{x^4 + 3x^2 - 4}{x-2}$

The binomial divisor is $x - 2$ so $k = 2$.

Add each column, multiply the result by 2, and repeat until the last column is reached.

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & 3 & 0 & -4 \\ & & 2 & 4 & 14 & 28 \\ \hline & 1 & 2 & 7 & 14 & 24 \end{array}$$

The quotient is $x^3 + 2x^2 + 7x + 14$. The remainder is 24.

Thus, $\dfrac{x^4 + 3x^2 - 4}{x-2} = x^3 + 2x^2 + 7x + 14 + \dfrac{26}{x-2}$

12. -

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

13. $f(x) = 2x^3 + 5x^2 - 6x - 9$

The Rational Zero Theorem tells us that if $\dfrac{p}{q}$ is a zero of $f(x)$, then p is a factor of –9 and q is a factor of 2.

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} = \frac{\text{factor of } -9}{\text{factor of } 2}$$

The factors of 2 are ± 1 and ± 2 and the factors of –9 are $\pm 1, \pm 3$ and ± 9. The possible values for $\dfrac{p}{q}$ are $\pm\dfrac{1}{2}, \pm 1, \pm\dfrac{3}{2}, \pm\dfrac{9}{2}$ and ± 9.

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –1.

$$\begin{array}{r|rrrr} -1 & 2 & 5 & -6 & -9 \\ & & -2 & -3 & 9 \\ \hline & 2 & 3 & -9 & 0 \end{array}$$

Dividing by $(x+1)$ gives a remainder of 0, so –1 is a zero of the function. The polynomial can be written as

$(x+1)(2x^2+3x-9).$

We can factor the quadratic factor to write the polynomial as

$(2x^2+3x-9)=(2x-3)(x+3)$

To find the other zeros, set each factor equal to 0.

$(2x-3)=0$ or $(x+3)=0$

$x=\dfrac{3}{2}$ $\qquad x=-3$

The zeros of $2x^3+5x^2-6x-9$ are $-3,-1,\dfrac{3}{2}.$

14. -

15. $f(x)=4x^4+16x^3+13x^2-15x-18$

The Rational Zero Theorem tells us that if $\dfrac{p}{q}$ is a zero of $f(x),$ then p is a factor of –18 and q is a factor of 4.

$$\frac{p}{q}=\frac{\text{factor of constant term}}{\text{factor of leading coefficient}}=\frac{\text{factor of }-18}{\text{factor of 4}}$$

The factors of 4 are $\pm1,\pm2$ and ±4 and the factors of –18 are $\pm1,\pm2,\pm3,\pm6,\pm9$ and $\pm18.$

The possible values for $\dfrac{p}{q}$ are $\pm1,\pm\dfrac{1}{2},\pm\dfrac{1}{4},\pm2,\pm3,\pm\dfrac{3}{2},\pm\dfrac{3}{4},\pm6,\pm9,\pm\dfrac{9}{2},\pm\dfrac{9}{4}$ and $\pm18.$

These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

$$\begin{array}{r|rrrrr} 1 & 4 & 16 & 13 & -15 & -18 \\ & & 4 & 20 & 33 & 18 \\ \hline & 4 & 20 & 33 & 18 & 0 \end{array}$$

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$(x-1)(4x^3+20x^2+33x+18)$

Again, use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's consider -2.

$$\begin{array}{r|rrrr} -2 & 4 & 20 & 33 & 18 \\ & & -8 & -24 & -18 \\ \hline & 4 & 12 & 9 & 0 \end{array}$$

Dividing by $(x+2)$ gives a remainder of 0, so -2 is a zero of the function. The polynomial can be written as

$(x-1)(x+2)(4x^2+12x+9)$.

Factor the quadratic equation.

$(4x^2+12x+9)=(2x+3)^2$

To find the other zeros, set the factor equal to 0.

$$(2x+3)^2=0$$
$$2x+3=0$$
$$x=-\frac{3}{2}$$

The zeros of $4x^4+16x^3+13x^2-15x-18$ are $1, -2,$ and $-\frac{3}{2}$ (multiplicity 2).

16. -

Given the following information about a polynomial function, find the function.

17. It has a double zero at $x=3$ and zeroes at $x=1$ and $x=-2$. It's *y*-intercept is $(0, 12)$.

The polynomial must have factors of $(x-3)^2, (x-1)$ and $(x+2)$.

Let's begin by multiplying these factors.

$f(x)=a(x-3)^2(x-1)(x+2)$

Substitute $x=0$ and $f(0)=12$ into $f(x)$.

$$12=a(0-3)^2(0-1)(0+2)$$
$$12=-18a$$
$$-\frac{12}{18}=a$$
$$-\frac{2}{3}=a$$

So the polynomial function is $f(x)=-\frac{2}{3}(x-3)^2(x-1)(x+2)$.

18. -

Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

19. $8x^3 - 21x^2 + 6 = 0$

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= 8(-x)^3 - 21(-x)^2 + 6 \\ &= -8x^3 - 21x^2 + 6 \end{aligned}$$

There is only one sign change, so there is 1 negative real root.

For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

20. -

21. $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 4}$

First, factor the numerator and denominator.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 3}{x^2 - 4} \\ &= \frac{(x-1)(x+3)}{(x+2)(x-2)} \end{aligned}$$

We can find the *y*-intercept by evaluating the function at zero.

$$f(0) = \frac{(0-1)(0+3)}{(0+2)(0-2)} = \frac{3}{4}$$

The *x*-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-1)(x+3)}{(x+2)(x-2)}$$ This is zero when the numerator is zero.

$$0 = (x-1)(x+3)$$

Set each factor equal to zero.

$$x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 1 \qquad\qquad x = -3$$

The y-intercept is $\left(0,\dfrac{3}{4}\right)$, the x-intercepts are $(-3,0),(1,0).$

At the x-intercept, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(x+2)(x-2)=0$$

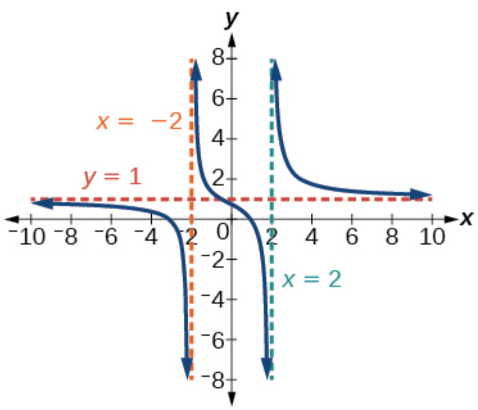
Set each factor equal to zero.

$$x+2=0 \quad \text{or} \quad x-2=0$$
$$x=-2 \qquad\qquad x=2$$

The vertical asymptotes are $x=2$ and $x=-2$.

Since the degree of the denominator = degree of the numerator we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y=\frac{1}{1}$ or $y=1$.

The graph of the given function is shown.



Find the slant asymptote of the rational function.

22. -

Find the inverse of the function.

23. $f(x)=\sqrt{x-2}+4$

The original function $f(x)=\sqrt{x-2}+4$ is one-to-one on its domain of $x\geq 2$.

Remember, the domain is found by noting that the radicand of an even index radical function must be non-negative.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = \sqrt{x-2} + 4$ Interchange x and y.

$x = \sqrt{y-2} + 4$ Subtract 4 from both sides.

$x - 4 = \sqrt{y-2}$ Square each side.

$(x-4)^2 = y - 2$ Add 2 to both sides.

$(x-4)^2 + 2 = y$ Rename the function $f^{-1}(x)$.

$f^{-1}(x) = (x-4)^2 + 2$

Note that the original function has range $f(x) \geq 4$.

domain of f^{-1} = range of $f = [4, \infty)$.

24. -

25. $f(x) = \dfrac{2x+3}{3x-1}$

This is a transformation of the basic rational function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$y = \dfrac{2x+3}{3x-1}$ Interchange x and y.

$x = \dfrac{2y+3}{3y-1}$ Multiply $3y - 1$ on both sides.

$x(3y-1) = 2y + 3$ Distribute x.

$3xy - x = 2y + 3$ Subtract $2y$ from both sides.

$3xy - x - 2y = 3$ Add x to both sides.

$3xy - 2y = x + 3$ Divide $3x - 2$ by both sides.

$y = \dfrac{x+3}{3x-2}$ Rename the function $f^{-1}(x)$.

$f^{-1}(x) = \dfrac{x+3}{3x-2}$

Find the unknown value.

26. -

27. y varies jointly with x and the cube root of z. If when $x = 2$ and $z = 27$, $y = 12$, find y if $x = 5$ and $z = 8$.

Begin by writing an equation to show the relationship between the variables.

$$y = kx\sqrt[3]{z}$$

Substitute $x = 2$, $z = 27$ and $y = 12$ to find the value of the constant k.

$$12 = k(2)\sqrt[3]{27}$$
$$12 = k(2)(3)$$
$$\frac{12}{6} = k$$
$$2 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$y = 2x\sqrt[3]{z}$$

Substitute $x = 5, z = 8$ and solve for y.

$$y = 2(5)\sqrt[3]{8} = 2(5)(2) = 20$$

Solve the following application problem.

28. -