Chapter 9
Trigonometric Identities and Equations
9.1 Solving Trigonometric Equations with Identities

Section Exercises
Verbal

1. We know $g(x) = \cos x$ is an even function, and $f(x) = \sin x$ and $h(x) = \tan x$ are odd functions. What about $G(x) = \cos^2 x$, $F(x) = \sin^2 x$, and $H(x) = \tan^2 x$? Are they even, odd, or neither? Why?

   All three functions, F, G, and H, are even.

   This is because $F(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x = F(x)$,

   $G(-x) = \cos(-x)\cos(-x) = \cos x \cos x = \cos^2 x = G(x)$ and

   $H(-x) = \tan(-x)\tan(-x) = (-\tan x)(-\tan x) = \tan^2 x = H(x)$.

2. -

3. After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

   When $\cos t = 0$, then $\sec t = \frac{1}{0}$, which is undefined.

4. -

Algebraic

For the following exercises, use the fundamental identities to fully simplify the expression.

5. $\sin x \cos x \sec x$

   $\sin x$

6. -

7. $\tan x \sin x + \sec x \cos^2 x$

   $\sec x$

8. -

9. $\dfrac{\cot t + \tan t}{\sec(-t)}$

   $\csc t$

10. -

11. $-\tan(-x)\cot(-x)$

    -1

12. -

13. $\dfrac{1+\tan^2\theta}{\csc^2\theta} + \sin^2\theta + \dfrac{1}{\sec^2\theta}$

    $\sec^2 x$

14. -

15. $\dfrac{1-\cos^2 x}{\tan^2 x}+2\sin^2 x$

$\sin^2 x+1$

For the following exercises, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

16. -

17. $\dfrac{\sec x+\csc x}{1+\tan x}$; $\sin x$

$\dfrac{1}{\sin x}$

18. -

19. $\dfrac{1}{\sin x\cos x}-\cot x$; $\cot x$

$\dfrac{1}{\cot x}$

20. -

21. $(\sec x+\csc x)(\sin x+\cos x)-2-\cot x$; $\tan x$

$\tan x$

22. -

23. $\dfrac{1-\sin x}{1+\sin x}-\dfrac{1+\sin x}{1-\sin x}$; $\sec x$ and $\tan x$

$-4\sec x\tan x$

24. -

25. $\sec x$; $\cot x$

$\pm\sqrt{\dfrac{1}{\cot^2 x}+1}$

26. -

27. $\cot x$; $\sin x$

$\dfrac{\pm\sqrt{1-\sin^2 x}}{\sin x}$

28. -

For the following exercises, verify the identity.

29. $\cos x-\cos^3 x=\cos x\sin^2 x$

Answers will vary. Sample proof:

$$\cos x-\cos^3 x=\cos x\left(1-\cos^2 x\right)$$
$$=\cos x\sin^2 x$$

30. –

31. $\dfrac{1+\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

Answers will vary. Sample proof:

$\dfrac{1+\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = \sec^2 x + \tan^2 x = \tan^2 x + 1 + \tan^2 x = 1 + 2\tan^2 x$

32. -

33. $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

Answers will vary. Sample proof:

$\cos^2 x - \tan^2 x = 1 - \sin^2 x - \left(\sec^2 x - 1\right) = 1 - \sin^2 x - \sec^2 x + 1 = 2 - \sin^2 x - \sec^2 x$

Extensions

For the following exercises, prove or disprove the identity.

34. -

35. $\csc^2 x\left(1+\sin^2 x\right) = \cot^2 x$

False

36. -

37. $\dfrac{\tan x}{\sec x}\sin\left(-x\right) = \cos^2 x$

False

38. -

39. $\dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1+\sin\left(-x\right)}$

Proved with negative and Pythagorean identities.

For the following exercises, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

40. -

41. $3\sin^2\theta + 4\cos^2\theta = 3 + \cos^2\theta$

True $3\sin^2\theta + 4\cos^2\theta = 3\sin^2\theta + 3\cos^2\theta + \cos^2\theta = 3\left(\sin^2\theta + \cos^2\theta\right) + \cos^2\theta = 3 + \cos^2\theta$

42. -

Chapter 9
Trigonometric Identities and Equations
9.2 Sum and Difference Identities

Section Exercises

Verbal

1. Explain the basis for the cofunction identities and when they apply.
   The cofunction identities apply to complementary angles. Viewing the two acute angles of a right triangle, if one of those angles measures $x,$ the second angle measures $\frac{\pi}{2} - x.$
   Then $\sin x = \cos(\frac{\pi}{2} - x).$ The same holds for the other cofunction identities. The key is that the angles are complementary.
2. -
3. Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x) = \sin(x)$ and $g(x) = \cos(x)$. (Hint: $0 - x = -x$)
   $\sin(-x) = -\sin x,$ so $\sin x$ is odd. $\cos(-x) = \cos(0 - x) = \cos x,$ so $\cos x$ is even.

Algebraic

For the following exercises, find the exact value.

4. –
5. $\cos\left(\frac{\pi}{12}\right)$
   $\frac{\sqrt{2}+\sqrt{6}}{4}$
6. –
7. $\sin\left(\frac{11\pi}{12}\right)$
   $\frac{\sqrt{6}-\sqrt{2}}{4}$
8. -
9. $\tan\left(\frac{19\pi}{12}\right)$
   $-2-\sqrt{3}$

For the following exercises, rewrite in terms of $\sin x$ and $\cos x$.

10. -

11. $\sin\left(x-\frac{3\pi}{4}\right)$

$-\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x$

12. -

13. $\cos\left(x+\frac{2\pi}{3}\right)$

$-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

For the following exercises, simplify the given expression.

14. -

15. $\sec\left(\frac{\pi}{2}-\theta\right)$

$\csc\theta$

16. -

17. $\tan\left(\frac{\pi}{2}-x\right)$

$\cot x$

18. -

19. $\dfrac{\tan\left(\frac{3}{2}x\right)-\tan\left(\frac{7}{5}x\right)}{1+\tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

$\tan\left(\frac{x}{10}\right)$

For the following exercises, find the requested information.

20. -

21. Given that $\sin a = \frac{4}{5}$, and $\cos b = \frac{1}{3}$, with a and b both in the interval $\left[0, \frac{\pi}{2}\right)$, find $\sin(a-b)$ and $\cos(a+b)$.

$$\sin(a-b) = \left(\frac{4}{5}\right)\left(\frac{1}{3}\right) - \left(\frac{3}{5}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{4-6\sqrt{2}}{15}$$

$$\cos(a+b) = \left(\frac{3}{5}\right)\left(\frac{1}{3}\right) - \left(\frac{4}{5}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{3-8\sqrt{2}}{15}$$

For the following exercises, find the exact value of each expression.

22. -

23. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

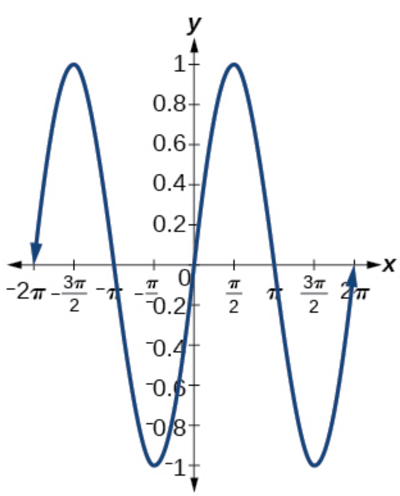
24. -

Graphical

For the following exercises, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

25. $\cos\left(\frac{\pi}{2} - x\right)$

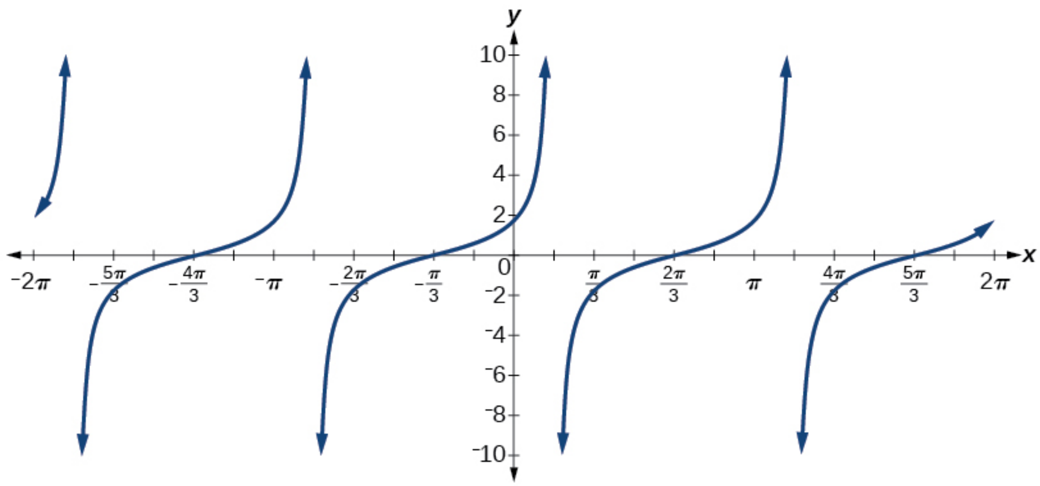
$\sin x$



26. -

27. $\tan\left(\frac{\pi}{3}+x\right)$

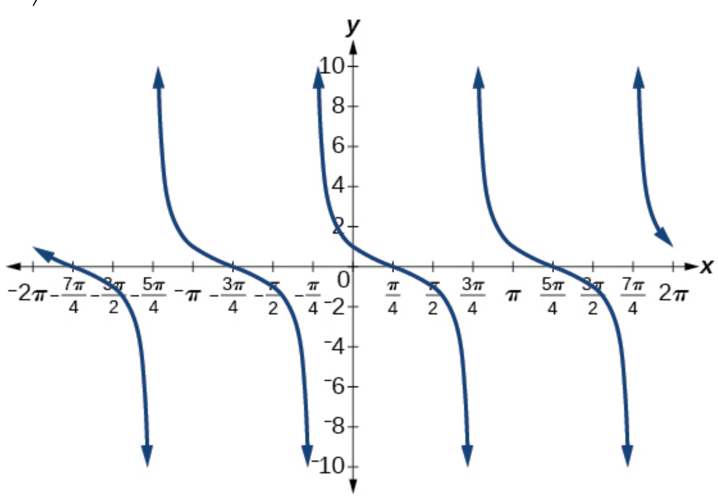
$\cot\left(\frac{\pi}{6}-x\right)$



28. -

29. $\tan\left(\frac{\pi}{4}-x\right)$

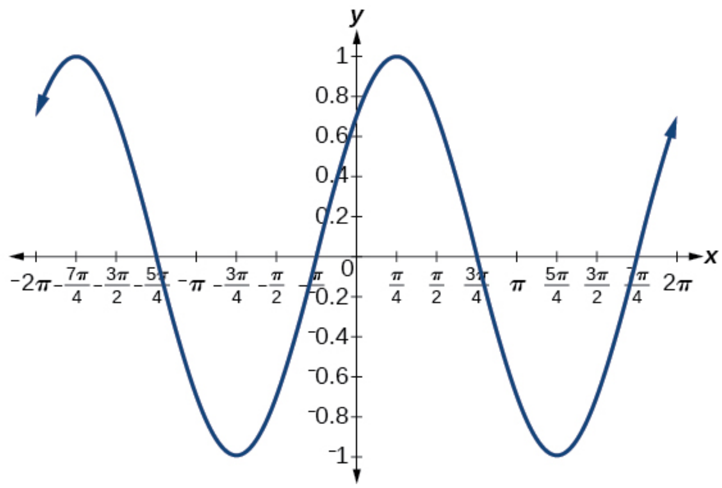
$\cot\left(\frac{\pi}{4}+x\right)$



30. -

31. $\sin\left(\frac{\pi}{4}+x\right)$

$$\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}$$



32. -

For the following exercises, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x = x + x$)

33. $f(x)=\sin(4x)-\sin(3x)\cos x,\ g(x)=\sin x\cos(3x)$

They are the same.

34. -

35. $f(x)=\sin(3x)\cos(6x),\ g(x)=-\sin(3x)\cos(6x)$

They are different, try $g(x)=\sin(9x)-\cos(3x)\sin(6x)$.

36. -

37. $f(x)=\sin(2x),\ g(x)=2\sin x\cos x$

They are the same.

38. -

39. $f(\theta)=\tan(2\theta), g(\theta)=\frac{\tan\theta}{1+\tan^2\theta}$

They are different, try $g(\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$.

40. -

41. $f(x)=\tan(-x),\ g(x)=\dfrac{\tan x-\tan(2x)}{1-\tan x\tan(2x)}$

They are different, try $g(x)=\dfrac{\tan x-\tan(2x)}{1+\tan x\tan(2x)}$.

Technology

For the following exercises, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

42. -

43. $\sin(195^{\circ})$

$-\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, or -0.2588

44. -

45. $\cos(345^{\circ})$

$\dfrac{1+\sqrt{3}}{2\sqrt{2}}$, or 0.9659

46. -

Extensions

For the following exercises, prove the identities provided.

47. $\tan\left(x+\dfrac{\pi}{4}\right)=\dfrac{\tan x+1}{1-\tan x}$

$$\tan\left(x+\frac{\pi}{4}\right)=$$

$$\frac{\tan x+\tan\left(\frac{\pi}{4}\right)}{1-\tan x\tan\left(\frac{\pi}{4}\right)}=$$

$$\frac{\tan x+1}{1-\tan x(1)}=\frac{\tan x+1}{1-\tan x}$$

48. -

49. $\dfrac{\cos(a+b)}{\cos a\cos b}=1-\tan a\tan b$

$$\frac{\cos(a+b)}{\cos a\cos b}=$$

$$\frac{\cos a\cos b}{\cos a\cos b}-\frac{\sin a\sin b}{\cos a\cos b}=1-\tan a\tan b$$

50. -

51. $\dfrac{\cos(x+h)-\cos x}{h}=\cos x\dfrac{\cos h-1}{h}-\sin x\dfrac{\sin h}{h}$

$$\frac{\cos(x+h)-\cos x}{h}=$$

$$\frac{\cos x\cosh-\sin x\sinh-\cos x}{h}=$$

$$\frac{\cos x(\cosh-1)-\sin x\sinh}{h}=\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}$$

For the following exercises, prove or disprove the statements.

52. -

53. $\tan(u-v)=\dfrac{\tan u-\tan v}{1+\tan u\tan v}$

True

54. -

55. If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta)=\sin\gamma$.

True. Note that $\sin(\alpha+\beta)=\sin(\pi-\gamma)$ and expand the right hand side.

56. -

Chapter 9
Trigonometric Identities and Equations
9.3 Double-Angle, Half-Angle, and Reduction Formulas

Section Exercises
Verbal

1. Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$.

   Use the Pythagorean identities and isolate the squared term.
2. -
3. We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \pm\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

   $\frac{1-\cos x}{\sin x}\ \frac{\sin x}{1+\cos x}$, multiplying the top and bottom by $\sqrt{1-\cos x}$ and $\sqrt{1+\cos x}$, respectively.
4. -

Algebraic

For the following exercises, find the exact values of a) sin2*x* b) cos2*x*, and c) tan2*x* without solving for *x*.

5. If $\sin x = \frac{1}{8}$, and *x* is in quadrant I.

   a) $\frac{3\sqrt{7}}{32}$ b) $\frac{31}{32}$ c) $\frac{3\sqrt{7}}{31}$
6. -
7. If $\cos x = -\frac{1}{2}$, and *x* is in quadrant III.

   a) $\frac{\sqrt{3}}{2}$ b) $-\frac{1}{2}$ c) $-\sqrt{3}$
8. -

For the following exercises, find the values of the six trigonometric functions if the conditions provided hold.

9. $\cos(2\theta) = \frac{3}{5}$ and $90^\circ \le \theta \le 180^\circ$

$\cos\theta = -\frac{2\sqrt{5}}{5}$, $\sin\theta = \frac{\sqrt{5}}{5}$, $\tan\theta = -\frac{1}{2}$, $\csc\theta = \sqrt{5}$, $\sec\theta = -\frac{\sqrt{5}}{2}$, $\cot\theta = -2$

10. -

For the following exercises, simplify to one trigonometric expression.

11. $2\sin\left(\frac{\pi}{4}\right)2\cos\left(\frac{\pi}{4}\right)$

$2\sin\left(\frac{\pi}{2}\right)$

12. -

For the following exercises, find the exact value using half-angle formulas.

13. $\sin\left(\frac{\pi}{8}\right)$

$\frac{\sqrt{2-\sqrt{2}}}{2}$

14. -

15. $\sin\left(\frac{11\pi}{12}\right)$

$\frac{\sqrt{2-\sqrt{3}}}{2}$

16. -]

17. $\tan\left(\frac{5\pi}{12}\right)$

$2+\sqrt{3}$

18. -

19. $\tan\left(-\frac{3\pi}{8}\right)$

$-1-\sqrt{2}$

For the following exercises, find the exact values of a) $\sin\left(\frac{x}{2}\right)$, b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$ without solving for x, when $0 \le x \le 360°$.

20. -

21. If $\sin x = -\frac{12}{13}$, and x is in quadrant III.

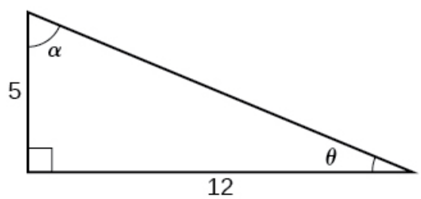
a) $\frac{3\sqrt{13}}{13}$ b) $-\frac{2\sqrt{13}}{13}$ c) $-\frac{3}{2}$

22. -

23. If $\sec x = -4$, and x is in quadrant II.

a) $\frac{\sqrt{10}}{4}$ b) $\frac{\sqrt{6}}{4}$ c) $\frac{\sqrt{15}}{3}$

For the following exercises, use the diagram to find the requested half and double angles.



24. -

25. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

$\frac{120}{169}, -\frac{119}{169}, -\frac{120}{119}$

26. -

27. Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

$\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, \frac{2}{3}$

For the following exercises, simplify each expression. Do not evaluate.

28. -

29. $2\cos^2(37^\circ) - 1$

$\cos(74^\circ)$

30. -

31. $\cos^2(9x) - \sin^2(9x)$

$\cos(18x)$

32. -

33. $6\sin(5x)\cos(5x)$

$3\sin(10x)$

For the following exercises, prove the identity given.

34. -

35. $\sin(2x) = -2\sin(-x)\cos(-x)$

$-2\sin(-x)\cos(-x) = -2(-\sin(x)\cos(x)) = \sin(2x)$

36. -

37. $\dfrac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \tan^3\theta$

$\dfrac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \dfrac{2\sin(\theta)\cos(\theta)}{1+\cos^2\theta - \sin^2\theta}\tan^2\theta =$

$\dfrac{2\sin(\theta)\cos(\theta)}{2\cos^2\theta}\tan^2\theta = \dfrac{\sin(\theta)}{\cos\theta}\tan^2\theta =$

$\cot(\theta)\tan^2\theta = \tan\theta$

For the following exercises, rewrite the expression with an exponent no higher than 1.

38. -

39. $\cos^2(6x)$

$\dfrac{1+\cos(12x)}{2}$

40. -

41. $\sin^4(3x)$

$\dfrac{3+\cos(12x)-4\cos(6x)}{8}$

42. -

43. $\cos^4 x\sin^2 x$

$\dfrac{2+\cos(2x)-2\cos(4x)-\cos(6x)}{32}$

44. -

Technology

For the following exercises, reduce the equations to powers of one, and then check the answer graphically.

45. $\tan^4 x$

$$\frac{3+\cos(4x)-4\cos(2x)}{3+\cos(4x)+4\cos(2x)}$$

46. -

47. $\sin^2 x\cos^2 x$

$$\frac{1-\cos(4x)}{8}$$

48. -

49. $\tan^4 x\cos^2 x$

$$\frac{3+\cos(4x)-4\cos(2x)}{4(\cos(2x)+1)}$$

50. -

51. $\cos^2(2x)\sin x$

$$\frac{\left(1+\cos(4x)\right)\sin x}{2}$$

52. -

For the following exercises, algebraically find an equivalent function, only in terms of $\sin x$ and/or $\cos x$, and then check the answer by graphing both equations.

53. $\sin(4x)$

$$4\sin x\cos x\left(\cos^2 x-\sin^2 x\right)$$

54. -

Extensions

For the following exercises, prove the identities.

55. $\sin(2x)=\dfrac{2\tan x}{1+\tan^2 x}$

$$\frac{2\tan x}{1+\tan^2 x}=\frac{\dfrac{2\sin x}{\cos x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}=\frac{\dfrac{2\sin x}{\cos x}}{\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}}=$$

$$\frac{2\sin x}{\cos x}.\frac{\cos^2 x}{1}=2\sin x\cos x=\sin(2x)$$

56. -

57. $\tan(2x) = \dfrac{2\sin x\cos x}{2\cos^2 x - 1}$

$$\frac{2\sin x\cos x}{2\cos^2 x - 1} = \frac{\sin(2x)}{\cos(2x)} = \tan(2x)$$

58. -

59. $\sin(3x) = 3\sin x\cos^2 x - \sin^3 x$

$$\begin{aligned}\sin(x+2x) &= \sin x\cos(2x) + \sin(2x)\cos x \\ &= \sin x(\cos^2 x - \sin^2 x) + 2\sin x\cos x\cos x \\ &= \sin x\cos^2 x - \sin^3 x + 2\sin x\cos^2 x \\ &= 3\sin x\cos^2 x - \sin^3 x\end{aligned}$$

60. -

61. $\dfrac{1+\cos(2t)}{\sin(2t) - \cos t} = \dfrac{2\cos t}{2\sin t - 1}$

$$\begin{aligned}\frac{1+\cos(2t)}{\sin(2t)-\cos t} &= \frac{1+2\cos^2 t - 1}{2\sin t\cos t - \cos t} \\ &= \frac{2\cos^2 t}{\cos t(2\sin t - 1)} \\ &= \frac{2\cos t}{2\sin t - 1}\end{aligned}$$

62. -

63. $\cos(16x) = \left(\cos^2(4x) - \sin^2(4x) - \sin(8x)\right)\left(\cos^2(4x) - \sin^2(4x) + \sin(8x)\right)$

$$\begin{aligned}&\left(\cos^2(4x) - \sin^2(4x) - \sin(8x)\right)\left(\cos^2(4x) - \sin^2(4x) + \sin(8x)\right) = \\ &= \left(\cos(8x) - \sin(8x)\right)\left(\cos(8x) + \sin(8x)\right) \\ &= \cos^2(8x) - \sin^2(8x) \\ &= \cos(16x)\end{aligned}$$

Chapter 9
Trigonometric Identities and Equations
9.4 Sum-to-Product and Product-to-Sum Formulas

Section Exercises
Verbal

1. Starting with the product to sum formula $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$, explain how to determine the formula for $\cos\alpha\sin\beta$.
   Substitute α into cosine and β into sine and evaluate.

2. -

3. Explain a situation where we would convert an equation from a sum to a product and give an example.
   Answers will vary. There are some equations that involve a sum of two trig expressions where when converted to a product are easier to solve. For example: $\frac{\sin(3x)+\sin x}{\cos x} = 1$.
   When converting the numerator to a product the equation becomes: $\frac{2\sin(2x)\cos x}{\cos x} = 1$.

4. -

Algebraic

For the following exercises, rewrite the product as a sum or difference.

5. $16\sin(16x)\sin(11x)$
   $8(\cos(5x)-\cos(27x))$
6. -
7. $2\sin(5x)\cos(3x)$
   $\sin(2x)+\sin(8x)$
8. -
9. $\sin(-x)\sin(5x)$
   $\frac{1}{2}(\cos(6x)-\cos(4x))$
10. -

For the following exercises, rewrite the sum or difference as a product.

11. $\cos(6t)+\cos(4t)$

    $2\cos(5t)\cos t$

12. -

13. $\cos(7x)+\cos(-7x)$

    $2\cos(7x)$

14. -

15. $\cos(3x)+\cos(9x)$

    $2\cos(6x)\cos(3x)$

16. -

For the following exercises, evaluate the product for the following using a sum or difference of two functions. Evaluate exactly.

17. $\cos(45°)\cos(15°)$

    $\frac{1}{4}\left(1+\sqrt{3}\right)$

18. -

19. $\sin(-345°)\sin(-15°)$

    $\frac{1}{4}\left(\sqrt{3}-2\right)$

20. -

21. $\sin(-45°)\sin(-15°)$

    $\frac{1}{4}\left(\sqrt{3}-1\right)$

For the following exercises, evaluate the product using a sum or difference of two functions. Leave in terms of sine and cosine.

22. -

23. $2\sin(100°)\sin(20°)$

    $\cos(80°)-\cos(120°)$

24. -

25. $\sin(213°)\cos(8°)$

$\frac{1}{2}\left(\sin(221°)+\sin(205°)\right)$

26. -

For the following exercises, rewrite the sum as a product of two functions. Leave in terms of sine and cosine.

27. $\sin(76°)+\sin(14°)$

$\sqrt{2}\cos(31°)$

28. -

29. $\sin(101°)-\sin(32°)$

$2\cos(66.5°)\sin(34.5°)$

30. -

31. $\sin(-1°)+\sin(-2°)$

$2\sin(-1.5°)\cos(0.5°)$

For the following exercises, prove the identity.

32. -

33. $4\sin(3x)\cos(4x)=2\sin(7x)-2\sin x$

$2\sin(7x)-2\sin x = 2\sin(4x+3x)-2\sin(4x-3x) =$

$2(\sin(4x)\cos(3x)+\sin(3x)\cos(4x))-2(\sin(4x)\cos(3x)-\sin(3x)\cos(4x)) =$

$2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))-2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x)) =$

$4\sin(3x)\cos(4x)$

34. -

35. $\sin x+\sin(3x)=4\sin x\cos^2 x$

$\sin x+\sin(3x)=2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{-2x}{2}\right)=$

$2\sin(2x)\cos x = 2(2\sin x\cos x)\cos x =$

$4\sin x\cos^2 x$

36. -

37. $2\tan x\cos(3x)=\sec x\left(\sin(4x)-\sin(2x)\right)$

$$2\tan x\cos(3x)=\frac{2\sin x\cos(3x)}{\cos x}=\frac{2(.5(\sin(4x)-\sin(2x)))}{\cos x}=$$

$$\frac{1}{\cos x}\left(\sin(4x)-\sin(2x)\right)=\sec x\left(\sin(4x)-\sin(2x)\right)$$

38. -

Numeric

For the following exercises, rewrite the sum as a product of two functions or the product as a sum of two functions. Give your answer in terms of sines and cosines. Then evaluate the final answer numerically, rounded to four decimal places.

39. $\cos(58^\circ)+\cos(12^\circ)$

$2\cos(35^\circ)\cos(23^\circ),\ 1.5081$

40. -

41. $\cos(44^\circ)-\cos(22^\circ)$

$-2\sin(33^\circ)\sin(11^\circ),\ -0.2078$

42. -

43. $\sin(-14^\circ)\sin(85^\circ)$

$\frac{1}{2}\left(\cos(99^\circ)-\cos(71^\circ)\right),\ -0.2410$

Technology

For the following exercises, algebraically determine whether each of the given expressions is a true identity. If it is not an identity, replace the right-hand side with an expression equivalent to the left side. Verify the results by graphing both expressions on a calculator.

44. -

45. $\dfrac{\cos(10\theta)+\cos(6\theta)}{\cos(6\theta)-\cos(10\theta)}=\cot(2\theta)\cot(8\theta)$

It is an identity.

46. -

47. $2\cos(2x)\cos x+\sin(2x)\sin x=2\sin x$

It is not an identity, but $2\cos^3 x$ is.

48. -

For the following exercises, simplify the expression to one term, then graph the original function and your simplified version to verify they are identical.

49. $\dfrac{\sin(9t)-\sin(3t)}{\cos(9t)+\cos(3t)}$

$\tan(3t)$

50. -

51. $\dfrac{\sin(3x)-\sin x}{\sin x}$

$2\cos(2x)$

52. -

53. $\sin x\cos(15x)-\cos x\sin(15x)$

$-\sin(14x)$

Extensions

For the following exercises, prove the following Sum-to-Product Formulas.

54. -

55. $\cos x+\cos y=2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

Start with $\cos x+\cos y$. Make a substitution and let $x=\alpha+\beta$ and let $y=\alpha-\beta$, so $\cos x+\cos y$ becomes $\cos(\alpha+\beta)+\cos(\alpha-\beta)=$

$\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta=$

$2\cos\alpha\cos\beta$

Since $x=\alpha+\beta$ and $y=\alpha-\beta$, we can solve for α and β in terms of *x* and *y* and substitute in for $2\cos\alpha\cos\beta$ and get $2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$.

For the following exercises, prove the identity.

56. -

57. $\dfrac{\cos(3x)+\cos x}{\cos(3x)-\cos x}=-\cot(2x)\cot x$

$$\frac{\cos(3x)+\cos x}{\cos(3x)-\cos x}=\frac{2\cos(2x)\cos x}{-2\sin(2x)\sin x}=-\cot(2x)\cot x$$

58. -

59. $\dfrac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)} = \tan y$

$$\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)} = \frac{-2\sin(3y)\sin(-y)}{2\sin(3y)\cos y} =$$

$$\frac{2\sin(3y)\sin(y)}{2\sin(3y)\cos y} = \tan y$$

60. -

61. $\cos x - \cos(3x) = 4\sin^2 x\cos x$

$$\cos x - \cos(3x) = -2\sin(2x)\sin(-x) =$$

$$2(2\sin x\cos x)\sin x = 4\sin^2 x\cos x$$

62. -

63. $\tan\left(\dfrac{\pi}{4}-t\right) = \dfrac{1-\tan t}{1+\tan t}$

$$\tan\left(\frac{\pi}{4}-t\right) = \frac{\tan\left(\frac{\pi}{4}\right)-\tan t}{1+\tan\left(\frac{\pi}{4}\right)\tan(t)} = \frac{1-\tan t}{1+\tan t}$$

Chapter 9
Trigonometric Identities and Equations
9.5 Solving Trigonometric Equations

SECTION EXERCISES

Verbal

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

   There will not always be solutions to trigonometric function equations. For a basic example, $\cos(x) = -5$.
2. -
3. When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

   If the sine or cosine function has a coefficient of one, isolate the term on one side of the equals sign. If the number it is set equal to has an absolute value less than or equal to one, the equation has solutions, otherwise it does not. If the sine or cosine does not have a coefficient equal to one, still isolate the term but then divide both sides of the equation by the leading coefficient. Then, if the number it is set equal to have an absolute value greater than one, the equation has no solution.

Algebraic

For the following exercises, find all solutions exactly on the interval $0 \le \theta < 2\pi$.

4. -
5. $2\sin\theta = \sqrt{3}$

   $\frac{\pi}{3}, \frac{2\pi}{3}$
6. -
7. $2\cos\theta = -\sqrt{2}$

   $\frac{3\pi}{4}, \frac{5\pi}{4}$
8. -
9. $\tan x = 1$

   $\frac{\pi}{4}, \frac{5\pi}{4}$
10. -

11. $4\sin^2 x - 2 = 0$

$\sin^2 x = \frac{1}{2}$

$\sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

12. -

For the following exercises, solve exactly on $[0, 2\pi)$.

13. $2\cos\theta = \sqrt{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\frac{\pi}{4}, \frac{7\pi}{4}$

14. -

15. $2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\frac{7\pi}{6}, \frac{11\pi}{6}$

16. -

17. $2\sin(3\theta) = 1$

$\sin(3\theta) = \frac{1}{2}$

$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

$\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

18. -

19. $2\cos(3\theta) = -\sqrt{2}$

$\cos(3\theta) = -\frac{\sqrt{2}}{2}$

$3\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \frac{21\pi}{4}$

$\frac{3\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}$

20. -

21. $2\sin(\pi\theta)=1$

$\sin(\pi\theta)=\frac{1}{2}$

$\pi\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6},\frac{37\pi}{6}$

$\frac{1}{6},\frac{5}{6},\frac{13}{6},\frac{17}{6},\frac{25}{6},\frac{29}{6},\frac{37}{6}$

22. -

For the following exercises, find all exact solutions on $\left[0,2\pi\right)$.

23. $\sec(x)\sin(x)-2\sin(x)=0$

$\frac{1}{\cos(x)}\sin(x)-2\sin(x)=0$

$\sin x=0$

$x=0,\pi$

$\frac{1}{\cos(x)}\sin(x)=2\sin(x)$

$\frac{1}{\cos(x)}=2$

$\cos(x)=\frac{1}{2}$

$x=\frac{\pi}{3},\frac{5\pi}{3}$

$0,\frac{\pi}{3},\pi,\frac{5\pi}{3}$

24. -

25. $2\cos^2 t+\cos\left(t\right)=1$

$2\cos^2 t+\cos\left(t\right)-1=0$

$(2\cos t-1)(\cos t+1)=0$

$\cos t=\dfrac{1}{2}$

$t=\dfrac{\pi}{3},\dfrac{5\pi}{3}$

$\cos t=-1$

$t=\pi$

$\dfrac{\pi}{3},\pi,\dfrac{5\pi}{3}$

26. -

27. $2\sin(x)\cos(x)-\sin(x)+2\cos(x)-1=0$

$\left(2\sin(x)\cos(x)-\sin(x)\right)+2\cos(x)-1=0$

$\sin\left(x\right)\left(2\cos(x)-1\right)+1\cdot\left(2\cos(x)-1\right)=0$

$\left(\sin\left(x\right)+1\right)\left(2\cos(x)-1\right)=0$

$\sin x=-1$

$x=\dfrac{3\pi}{2}$

$\cos x=\dfrac{1}{2}$

$x=\dfrac{\pi}{3},\dfrac{5\pi}{3}$

$\dfrac{\pi}{3},\dfrac{3\pi}{2},\dfrac{5\pi}{3}$

28. -

29. $\sec^2 x=1$

$\dfrac{1}{\cos^2 x}=1$

$\cos^2 x=1$

$\cos x=\pm\sqrt{1}=\pm 1$

$0,\pi$

30. -

31. $8\sin^2(x)+6\sin(x)+1=0$

$$8\sin^2(x)+6\sin(x)+1=0$$
$$(4\sin x+1)(2\sin x+1)=0$$
$$\sin x=-\frac{1}{4}$$
$$x=\sin^{-1}\left(-\frac{1}{4}\right)$$
$$\sin x=-\frac{1}{2}$$
$$x=\frac{7\pi}{6},\frac{11\pi}{6}$$
$$\pi-\sin^{-1}\left(-\frac{1}{4}\right),\frac{7\pi}{6},\frac{11\pi}{6},2\pi+\sin^{-1}\left(-\frac{1}{4}\right)$$

32. -

For the following exercises, solve with the methods shown in this section exactly on the interval $[0,2\pi)$.

33. $\sin(3x)\cos(6x)-\cos(3x)\sin(6x)=-0.9$

$$\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{\pi}{3}-\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{2\pi}{3}+\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\pi-\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),$$
$$\frac{4\pi}{3}+\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{5\pi}{3}-\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right)$$

34. -

35. $\cos(2x)\cos x+\sin(2x)\sin x=1$

$$(1-2\sin^2 x)\cos x+2\sin x\cos x\sin x=1$$
$$\cos x-2\sin^2 x\cos x+2\sin^2 x\cos x=1$$
$$\cos x=1$$
$$0$$

36. -

37. $9\cos(2\theta)=9\cos^2\theta-4$

$9(\cos^2\theta - \sin^2 x) = 9\cos^2\theta - 4$

$9\cos^2\theta - 9\sin^2 x = 9\cos^2\theta - 4$

$-9\sin^2\theta = -4$

$\sin^2\theta = \frac{4}{9}$

$\sin\theta = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$

$\theta = \sin^{-1}\left(\pm\frac{2}{3}\right)$

$\theta = \sin^{-1}\left(\frac{2}{3}\right), \pi - \sin^{-1}\left(\frac{2}{3}\right), \pi + \sin^{-1}\left(\frac{2}{3}\right), 2\pi - \sin^{-1}\left(\frac{2}{3}\right)$

38. -

39. $\cos(2t) = \sin t$

$1 - 2\sin^2 t = \sin t$

$0 = 2\sin^2 t + \sin t - 1$

$0 = (2\sin t - 1)(\sin t + 1)$

$\sin t = \frac{1}{2}$

$t = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin t = -1$

$t = \frac{3\pi}{2}$

$\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

40. -

For the following exercises, solve exactly on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.

41. $\tan^2 x - \sqrt{3}\tan x = 0$

$\tan^2 x - \sqrt{3}\tan x = 0$

$\tan x\left(\tan x - \sqrt{3}\right) = 0$

$\tan x = 0$

$x = 0, \pi$

$\tan x = \sqrt{3}$

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

$0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$

42. -

43. $\sin^2 x - 2\sin x - 4 = 0$

$\sin^2 x - 2\sin x - 4 = 0$

$\sin x = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 1\cdot(-4)}}{2\cdot 1} = \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$

no solutions, $1 \pm \sqrt{5}$ exceeds ± 1

There are no solutions.

44. -

45. $3\cos^2 x - 2\cos x - 2 = 0$

$\cos x = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 3\cdot(-2)}}{2\cdot 3} = \frac{2 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$

$x = \cos^{-1}\left(\frac{1 \pm \sqrt{7}}{3}\right)$

$\cos^{-1}\left(\frac{1}{3}\left(1 - \sqrt{7}\right)\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\left(1 - \sqrt{7}\right)\right)$

46. -

47. $\tan^2 x + 5\tan x - 1 = 0$

$$\tan x = \frac{-5 \pm \sqrt{(5)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \frac{-5 \pm \sqrt{29}}{2}$$

$$x = \tan^{-1}\left(\frac{-5 \pm \sqrt{29}}{2}\right)$$

$$\tan^{-1}\left(\frac{1}{2}\left(\sqrt{29}-5\right)\right), \pi + \tan^{-1}\left(\frac{1}{2}\left(-\sqrt{29}-5\right)\right), \pi + \tan^{-1}\left(\frac{1}{2}\left(\sqrt{29}-5\right)\right), 2\pi + \tan^{-1}\left(\frac{1}{2}\left(-\sqrt{29}-5\right)\right)$$

48. -

49. $-\tan^2 x - \tan x - 2 = 0$

$$\tan^2 x + \tan x + 2 = 0$$

$$\tan x = \frac{-1 \pm \sqrt{(1)^2 - 4\cdot 1\cdot(2)}}{2\cdot 1} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$$

There are no solutions.

For the following exercises, find exact solutions on the interval $[0, 2\pi)$. Look for opportunities to use trigonometric identities.

50. -

51. $\sin^2 x + \cos^2 x = 0$

$1 \neq 0$

There are no solutions

52. -

53. $\cos(2x) - \cos x = 0$

$$2\cos^2 x - 1 - \cos x = 0$$

$$2\cos^2 x - \cos x - 1 = 0$$

$$(2\cos x + 1)(\cos x - 1) = 0$$

$$\cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\cos x = 1$$

$$x = 0$$

$$0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

54. -

55. $1-\cos(2x)=1+\cos(2x)$

$0 = 2\cos(2x)$

$0 = \cos 2x$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

56. -

57. $10\sin x\cos x = 6\cos x$

$10\sin x\cos x - 6\cos x = 0$

$2\cos x(5\sin x - 3) = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x = \frac{3}{5}$

$x = \sin^{-1}\left(\frac{3}{5}\right)$

$\sin^{-1}\left(\frac{3}{5}\right), \frac{\pi}{2}, \pi - \sin^{-1}\left(\frac{3}{5}\right), \frac{3\pi}{2}$

58. -

59. $4\cos^2 x - 4 = 15\cos x$

$4\cos^2 x - 15\cos x - 4 = 0$

$(4\cos x + 1)(\cos x - 4) = 0$

$\cos x = -\frac{1}{4}$

$x = \cos^{-1}\left(-\frac{1}{4}\right)$

$\cos x = 4$

impossible

$\cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right)$

60. -

61. $8\cos^2\theta = 3 - 2\cos\theta$

$8\cos^2\theta + 2\cos\theta - 3 = 0$

$(2\cos\theta - 1)(4\cos\theta + 3) = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\cos\theta = -\frac{3}{4}$

$\theta = \cos^{-1}\left(-\frac{3}{4}\right)$

$\frac{\pi}{3}, \cos^{-1}\left(-\frac{3}{4}\right), 2\pi - \cos^{-1}\left(-\frac{3}{4}\right), \frac{5\pi}{3}$

62. -

63. $12\sin^2 t + \cos t - 6 = 0$

$12\left(1 - \cos^2 t\right) + \cos t - 6 = 0$

$12 - 12\cos^2 t + \cos t - 6 = 0$

$-12\cos^2 t + \cos t + 6 = 0$

$12\cos^2 t - \cos t - 6 = 0$

$(4\cos t - 3)(3\cos t + 2) = 0$

$\cos t = \frac{3}{4}$

$t = \cos^{-1}\left(\frac{3}{4}\right)$

$\cos t = -\frac{2}{3}$

$t = \cos^{-1}\left(-\frac{2}{3}\right)$

$\cos^{-1}\left(\frac{3}{4}\right), \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(\frac{3}{4}\right)$

64. -

65. $\cos^3 t = \cos t$

$\cos^3 t - \cos t = 0$

$\cos t(\cos^2 t - 1) = 0$

$\cos t(\cos t + 1)(\cos t - 1) = 0$

$\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos t = -1$

$t = \pi$

$\cos t = 1$

$t = 0$

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Graphical

For the following exercises, algebraically determine all solutions of the trigonometric equation exactly, then verify the results by graphing the equation and finding the zeros.

66. -

67. $8\cos^2 x - 2\cos x - 1 = 0$

$(2\cos x - 1)(4\cos x + 1) = 0$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$\cos x = -\frac{1}{4}$

$x = \cos^{-1}\left(-\frac{1}{4}\right)$

$\frac{\pi}{3}, \cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right), \frac{5\pi}{3}$

68. -

69. $2\cos^2 x - \cos x + 15 = 0$

$(2\cos x - 5)(\cos x - 3) = 0$

$\cos x = \frac{5}{2}$

$\cos x = 3$

There are no solutions.

70. -

71.

$(2\tan x + 3)(\tan x + 2) = 0$

$\tan x = -\frac{3}{2}$

$x = \tan^{-1}\left(-\frac{3}{2}\right)$

$\tan x = -2$

$x = \tan^{-1}(-2)$

$\pi + \tan^{-1}(-2), \pi + \tan^{-1}\left(-\frac{3}{2}\right), 2\pi + \tan^{-1}(-2), 2\pi + \tan^{-1}\left(-\frac{3}{2}\right)$

72. -

Technology

For the following exercises, use a calculator to find all solutions to four decimal places.

73. $\sin x = 0.27$

$2\pi k + 0.2734, 2\pi k + 2.8682$

74. -

75. $\tan x = -0.34$

$\pi k - 0.3277$

76. -

For the following exercises, solve the equations algebraically, and then use a calculator to find the values on the interval $[0, 2\pi)$. Round to four decimal places.

77. $\tan^2 x + 3\tan x - 3 = 0$

$$\tan x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{-3 \pm \sqrt{21}}{2} \approx 0.6694, 1.8287$$

Add π to 0.6694 and 1.8287 to obtain solutions in quadrants III & IV

$0.6694, 1.8287, 3.8110, 4.9703$

78. -

79. $\tan^2 x - \sec x = 1$

$$\frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos x} = 1$$

$$\cos^2 x \left(\frac{\sin^2 x}{\cos^2 x} - \frac{1}{\cos x} \right) = 1 \cdot \cos^2 x$$

$$\sin^2 x - \cos x - \cos^2 x = 0$$

$$\left(1 - \cos^2 x\right) - \cos x - \cos^2 x = 0$$

$$1 - 2\cos^2 x - \cos x = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$\left(2\cos x - 1\right)\left(\cos x + 1\right) = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\cos x = -1$$

$$x = \pi$$

$1.0472, 3.1416, 5.2360$

80. -

81. $2\tan^2 x + 9\tan x - 6 = 0$

$$\tan x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-6)}}{2 \cdot 2} = \frac{-9 \pm \sqrt{129}}{4}$$

Add π to values found in quadrants I & II to obtain solutions in quadrants III & IV

$0.5326, 1.7648, 3.6742, 4.9064$

82. -

Extensions

For the following exercises, find all solutions exactly to the equations on the interval $[0, 2\pi)$.

83. $\csc^2 x - 3\csc x - 4 = 0$

$(\csc x - 4)(\csc x + 1) = 0$

$\csc x = 4$

$\frac{1}{\sin x} = 4$

$\sin x = \frac{1}{4}$

$x = \sin^{-1}\left(\frac{1}{4}\right)$

$\csc x = -1$

$\sin x = -1$

$x = \frac{3\pi}{2}$

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right), \frac{3\pi}{2}$

84. -

85. $\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$

$\sin^2 x \cos^2 x + \cos^4 x = 0$

$\cos^2 x(\sin^2 x + \cos^2 x) = 0$

$\cos^2 x \cdot 1 = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\frac{\pi}{2}, \frac{3\pi}{2}$

86. -

87. $\sin^2 x - 1 + 2\cos(2x) - \cos^2 x = 1$

$-\cos^2 x + 2(2\cos^2 x - 1) - \cos^2 x = 1$

$2\cos^2 x = 3$

$\cos^2 x = \frac{3}{2}$

$\cos x = \pm\sqrt{\frac{3}{2}}$ the two values obtained are outside of the possible range

of values for cosine

There are no solutions.

88. -

89. $\frac{\sin(2x)}{\sec^2 x} = 0$

$\frac{2\sin x \cos x}{\frac{1}{\cos^2 x}} = 0$

$2\sin x \cos^3 x = 0$

$\sin x = 0$

$x = 0, \pi$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

90. -

91. $2\cos^2 x - \sin^2 x - \cos x - 5 = 0$

$2\cos^2 x - (1 - \cos^2 x) - \cos x - 5 = 0$

$3\cos^2 x - \cos x - 6 = 0$

$\cos x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-6)}}{2 \cdot 3} = \frac{1 \pm \sqrt{72}}{6} = \frac{1 \pm 6\sqrt{2}}{6}$

The two values obtained are outside of the possible range

of values for cosine

There are no solutions.

92. -

Real-World Applications

93. An airplane has only enough gas to fly to a city 200 miles northeast of its current location. If the pilot knows that the city is 25 miles north, how many degrees north of east should the airplane fly?

$\sin\theta = \frac{25}{200} = \frac{1}{8}$

$\theta = \sin^{-1}\left(\frac{1}{8}\right)$

7.2°

94. -

95. If a loading ramp is placed next to a truck, at a height of 2 feet, and the ramp is 20 feet long, what angle does the ramp make with the ground?

$\sin\theta = \frac{2}{20} = \frac{1}{10}$

$\theta = \sin^{-1}\left(\frac{1}{10}\right)$

5.7°

96. -

97. An astronaut is in a launched rocket currently 15 miles in altitude. If a man is standing 2 miles from the launch pad, at what angle is she looking down at him from horizontal? (Hint: this is called the angle of depression.)

$\tan\theta = \frac{15}{2}$

$\theta = \tan^{-1}\left(\frac{15}{2}\right)$

82.4°

98. -

99. A man is standing 10 meters away from a 6-meter tall building. Someone at the top of the building is looking down at him. At what angle is the person looking at him?

$\tan\theta = \frac{6}{10} = \frac{3}{5}$

$\theta = \tan^{-1}\left(\frac{3}{5}\right)$

31.0°

100. -

101. A 90-foot tall building has a shadow that is 2 feet long. What is the angle of elevation of the sun?

$\tan\theta = \frac{90}{2} = 45$

$\theta = \tan^{-1}(45)$

88.7°

102. -

103. A spotlight on the ground 3 feet from a 5-foot tall woman casts a 15-foot tall shadow on a wall 6 feet from the woman. At what angle is the light?

$\tan\theta = \frac{15}{(3+6)} = \frac{5}{3}$

$\theta = \tan^{-1}\left(\frac{5}{3}\right)$

59.0°

For the following exercises, find a solution to the following word problem algebraically. Then use a calculator to verify the result. Round the answer to the nearest tenth of a degree.

104. -

105. A person does a handstand with her feet touching a wall and her hands 3 feet away from the wall. If the person is 5 feet tall, what angle do her feet make with the wall?

$\sin\theta = \frac{3}{5}$

$\theta = \sin^{-1}\left(\frac{3}{5}\right)$

36.9°

106. -

Chapter 9 Review Exercises
Section 9.1

For the following exercises, find all solutions exactly that exist on the interval $[0,2\pi)$.

1. $\csc^2 t = 3$

   $\sin^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi - \sin^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi + \sin^{-1}\left(\frac{\sqrt{3}}{3}\right), 2\pi - \sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$
2. -
3. $2\sin\theta = -1$

   $\frac{7\pi}{6}, \frac{11\pi}{6}$
4. -
5. $9\sin\omega - 2 = 4\sin^2\omega$

   $\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right)$
6. -

For the following exercises, use basic identities to simplify the expression.

7. $\sec x \cos x + \cos x - \frac{1}{\sec x}$

   1
8. -

For the following exercises, determine if the given identities are equivalent.

9. $\sin^2 x + \sec^2 x - 1 = \frac{\left(1-\cos^2 x\right)\left(1+\cos^2 x\right)}{\cos^2 x}$

   Yes
10. -

Section 9.2

For the following exercises, find the exact value.

11. $\tan\left(\frac{7\pi}{12}\right)$

    $-2-\sqrt{3}$
12. -

13. $\sin\left(70^\circ\right)\cos\left(25^\circ\right)-\cos\left(70^\circ\right)\sin\left(25^\circ\right)$

$$\frac{\sqrt{2}}{2}$$

14. -

For the following exercises, prove the identity.

15. $\cos\left(4x\right)-\cos\left(3x\right)\cos x=\sin^2 x-4\cos^2 x\sin^2 x$

$$\begin{aligned}
\cos\left(4x\right)-\cos\left(3x\right)\cos x &= \cos\left(2x+2x\right)-\cos\left(x+2x\right)\cos x \\
&= \cos\left(2x\right)\cos(2x)-\sin(2x)\sin(2x)-\cos x\cos\left(2x\right)\cos x+\sin x\sin(2x)\cos x \\
&= \left(\cos^2 x-\sin^2 x\right)^2-4\cos^2 x\sin^2 x-\cos^2 x\left(\cos^2 x-\sin^2 x\right)+\sin x(2)\sin x\cos x\cos x \\
&= \left(\cos^2 x-\sin^2 x\right)^2-4\cos^2 x\sin^2 x-\cos^2 x\left(\cos^2 x-\sin^2 x\right)+2\sin^2 x\cos^2 x \\
&= \cos^4 x-2\cos^2 x\sin^2 x+\sin^4 x-4\cos^2 x\sin^2 x-\cos^4 x+\cos^2 x\sin^2 x+2\sin^2 x\cos^2 x \\
&= \sin^4 x-4\cos^2 x\sin^2 x+\cos^2 x\sin^2 x \\
&= \sin^2 x\left(\sin^2 x+\cos^2 x\right)-4\cos^2 x\sin^2 x \\
&= \sin^2 x-4\cos^2 x\sin^2 x
\end{aligned}$$

16. -

For the following exercise, simplify the expression.

17. $\dfrac{\tan\left(\frac{1}{2}x\right)+\tan\left(\frac{1}{8}x\right)}{1-\tan\left(\frac{1}{8}x\right)\tan\left(\frac{1}{2}x\right)}$

$$\tan\left(\frac{5}{8}x\right)$$

For the following exercises, find the exact value.

18. -

19. $\tan\left(\sin^{-1}\left(0\right)+\sin^{-1}\left(\frac{1}{2}\right)\right)$

$$\frac{\sqrt{3}}{3}$$

Section 9.3

For the following exercises, find the exact value.

20. -

21. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\sec\theta = -\frac{5}{3}$ and θ is in the interval $\left[\frac{\pi}{2}, \pi\right]$.

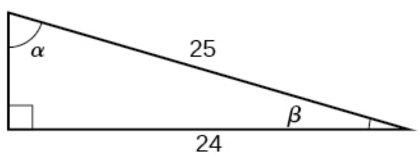
$-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}$

22. -

23. $\sec\left(\frac{3\pi}{8}\right)$

$\sqrt{2\left(2+\sqrt{2}\right)}$

For the following exercises, use the figure to find the desired quantities.



24. -

25. $\sin\left(\frac{\beta}{2}\right)$, $\cos\left(\frac{\beta}{2}\right)$, $\tan\left(\frac{\beta}{2}\right)$, $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$

$\frac{\sqrt{2}}{10}, \frac{7\sqrt{2}}{10}, \frac{1}{7}, \frac{3}{5}, \frac{4}{5}, \frac{3}{4}$

For the following exercises, prove the identity.

26. -

27. $\cot x \cos(2x) = -\sin(2x) + \cot x$

$$\begin{aligned}
\cot x \cos(2x) &= \cot x\left(1 - 2\sin^2 x\right) \\
&= \cot x - \frac{\cos x}{\sin x}(2)\sin^2 x \\
&= -2\sin x \cos x + \cot x \\
&= -\sin(2x) + \cot x
\end{aligned}$$

For the following exercises, rewrite the expression with no powers.

28. -

29. $\tan^2 x \sin^3 x$

$$\frac{10\sin x - 5\sin(3x) + \sin(5x)}{8(\cos(2x)+1)}$$

Section 9.4

For the following exercises, evaluate the product for the given expression using a sum or difference of two functions. Write the exact answer.

30. -

31. $2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{5\pi}{6}\right)$

$$\frac{\sqrt{3}}{2}$$

32. -

For the following exercises, evaluate the sum by using a product formula. Write the exact answer.

33. $\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{7\pi}{12}\right)$

$$-\frac{\sqrt{2}}{2}$$

34. -

For the following exercises, change the functions from a product to a sum or a sum to a product.

35. $\sin(9x)\cos(3x)$

$$\frac{1}{2}(\sin(6x) + \sin(12x))$$

36. -

37. $\sin(11x)+\sin(2x)$

$$2\sin\left(\frac{13}{2}x\right)\cos\left(\frac{9}{2}x\right)$$

38. -

Section 9.5

For the following exercises, find all exact solutions on the interval $[0,2\pi)$.

39. $\tan x+1=0$

$$\frac{3\pi}{4},\ \frac{7\pi}{4}$$

40. -

For the following exercises, find all exact solutions on the interval $[0,2\pi)$.

41. $2\sin^2 x-\sin x=0$

$$0,\frac{\pi}{6},\frac{5\pi}{6},\pi$$

42. -

43. $2\sin^2 x+5\sin x+3=0$

$$\frac{3\pi}{2}$$

44. -

45. $\frac{1}{\sec^2 x}+2+\sin^2 x+4\cos^2 x=0$

No solution

For the following exercises, simplify the equation algebraically as much as possible. Then use a calculator to find the solutions on the interval $[0,2\pi)$. Round to four decimal places.

46. -

47. $\csc^2 x-3\csc x-4=0$

$0.2527, 2.8889, 4.7124$

For the following exercises, graph each side of the equation to find the zeroes on the interval $[0,2\pi)$.

48. -

49. $\sec^2 x-2\sec x=15$

$1.3694,\ 1.9106,\ 4.3726,\ 4.9137$

Chapter 9 Practice Test

For the following exercises, simplify the given expression.

1. $\cos(-x)\sin x\cot x+\sin^2 x$

   1

2. -
3. $\csc(\theta)\cot(\theta)(\sec^2\theta-1)$

   $\sec(\theta)$

4. -

For the following exercises, find the exact value.

5. $\cos\left(\frac{7\pi}{12}\right)$

   $\frac{\sqrt{2}-\sqrt{6}}{4}$

6. -
7. $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\tan^{-1}\sqrt{3}\right)$

   $-\sqrt{2}-\sqrt{3}$

8. -
9. $\cos\left(\frac{3\pi}{4}+\theta\right)$

   $-\frac{1}{2}\cos(\theta)-\frac{\sqrt{3}}{2}\sin(\theta)$

10. -

For the following exercises, find all exact solutions to the equation on $[0,2\pi)$.

11. $\cos^2(32^\circ)\tan^2(32^\circ)$

    $\frac{1-\cos(64^\circ)}{2}$

12. -
13. $\cos^2 x-\sin^2 x-1=0$

    $0,\pi$

14. -

15. $\cos(2x)+\sin^2 x=0$

$\frac{\pi}{2},\frac{3\pi}{2}$

16. -

17. Rewrite the expression as a product instead of a sum: $\cos(2x)+\cos(-8x)$.

$2\cos(3x)\cos(5x)$

For the following exercises, rewrite the product as a sum or difference.

18. -

For the following exercises, rewrite the sum or differnce as a product.

19. $2\left(\sin(8\theta)-\sin(4\theta)\right)$

$4\sin(2\theta)\cos(6\theta)$

20. -

21. Find the solutions of $\sec^2 x-2\sec x=15$ on the interval $[0,2\pi)$ algebraically; then graph both sides of the equation to determine the answer.

1.3694, 1.9106, 4.3726, 4.9137

For the following exercises, find all solutions exactly on the interval $0\le\theta\le\pi$.

22. -

23. $\sqrt{3}\cot(y)=1$

60°

24. -

25. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta=\frac{7}{25}$ and θ is in quadrant IV.

$\frac{3}{5},-\frac{4}{5},-\frac{3}{4}$

26. -

For the following exercises, prove the identity.

27. $\tan^3 x-\tan x\sec^2 x=\tan(-x)$

$$\begin{aligned}
\tan^3 x-\tan x\sec^2 x &=\\
\tan x\left(\tan^2 x-\sec^2 x\right) &=\\
\tan x\left(\tan^2 x-\left(1+\tan^2 x\right)\right) &=\\
\tan x\left(\tan^2 x-1-\tan^2 x\right) &=\\
-\tan x &=\\
\tan(-x) &= \tan(-x)
\end{aligned}$$

28. -

29. $\frac{\sin(2x)}{\sin x}-\frac{\cos(2x)}{\cos x}=\sec x$

$$\frac{\sin(2x)}{\sin x}-\frac{\cos(2x)}{\cos x}=$$

$$\frac{2\sin x\cos x}{\sin x}-\frac{2\cos^2 x-1}{\cos x}=$$

$$2\cos x-2\cos x+\frac{1}{\cos x}=$$

$$\frac{1}{\cos x}=$$

$$\sec x=\sec x$$

30. -

31. The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t)=\frac{1}{4}\sin(120\pi t),$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

Amplitude: $\frac{1}{4}$, period $\frac{1}{60}$, frequency: 60 Hz

32. -

33. Two frequencies of sound are played on an instrument governed by the equation $n(t)=8\cos(20\pi t)\cos(1000\pi t)$. What are the period and frequency of the "fast" and "slow" oscillations? What is the amplitude?

Amplitude: 8, fast period: $\frac{1}{500}$, fast frequency: 500 Hz, slow period: $\frac{1}{10}$, slow frequency: 10 Hz

34. -

35. A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

$D(t)=20(0.9086)^t\cos(4\pi t)$, 31 seconds

36. -