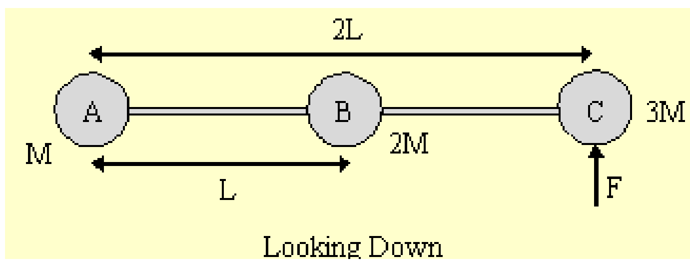
Physics 1120: Rotational Dynamics Solutions

Pulleys

1. Three point masses lying on a flat frictionless surface are connected by massless rods. Determine the angular acceleration of the body (a) about an axis through point mass A and out of the surface and (b) about an axis through point mass B. Express your answers in terms of F, L, and M. You will need to calculate the moment of inertia in each case.



First we will calculate the moments of inertia. Since these are point masses we use the formula $I = \Sigma m_i(r_i)^2$:

(a) $I_A = M(0)^2 + 2M(L)^2 + 3M(2L)^2 = 14ML^2$;

(b) $I_b = M(L)^2 + 2M(0)^2 + 3M(L)^2 = 4ML^2$.

The angular acceleration is governed by the rotational form of Newton's Second Law, $\Sigma\tau_z = I_z\alpha_z$, where z is out of the paper in this problem and τ_z, I_z, and α_z are all determined relative to the same axis.

	Axis A	***Axis B***
τ_z	2LF	LF
I_z	$14ML^2$	$4ML^2$
$\Sigma\tau_z = I_z\alpha_z$	$2LF = 14ML^2\alpha_A$	$LF = 4ML^2\alpha_B$

So the acceleration about axis A is

$$\alpha_A = F / 7ML ,$$

and the acceleration about axis B is

$$\alpha_B = F / 4ML .$$

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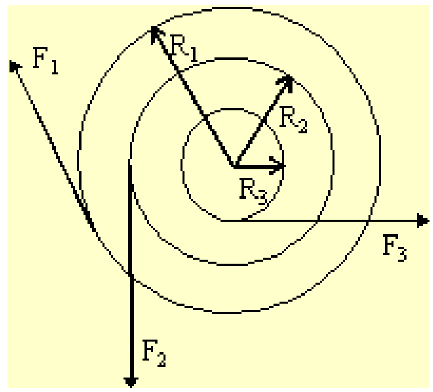
2. The object in the diagram below is on a fixed frictionless axle. It has a moment of inertia of $I = 50$ kg-m^2. The forces acting on the object are $F_1 = 100$ N, $F_2 = 200$ N, and $F_3 = 250$ N acting at different radii $R_1 = 60$ cm, $R_2 = 42$ cm, and $R_3 = 28$ cm. Find the angular acceleration of the object.



Since the axle is fixed we only need to consider the torques and use $\Sigma\tau_z = I_z\alpha_z$. Each of the forces is tangential to the object, i.e R and F are at 90º to one another. Recall that clockwise torques are negative or into the paper in this case.

$$\Sigma\tau_z = I_z\alpha_z$$

$$-R_1F_1 + R_2F_2 + R_3F_3 = I\alpha$$

So our equation for the acceleration is

$$\alpha = [-R_1F_1 + R_2F_2 + R_3F_3] / I .$$

Substituting in the given values, $\alpha = 1.88$ rad/s^2.

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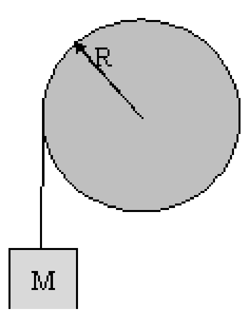
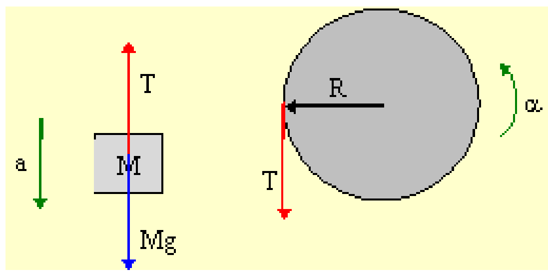
3. A rope is wrapped around a solid cylindrical drum. The drum has a fixed frictionless axle. The mass of the drum is 125 kg and it has a radius of $R = 50.0$ cm. The other end of the rope is tied to a block, $M = 10.0$ kg. What is the angular acceleration of the drum? What is the linear acceleration of the block? What is the tension in the rope? Assume that the rope does not slip.



Since the problem wants accelerations and forces, and one object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for each object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the drum has a fixed axle we need only consider the torques acting on it. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the block are weight and tension. Presumably the block will accelerate downwards. The only force directly acting on the drum which creates a torque is tension. Note that ropes, and therefore tensions, are always tangential to the object and thus normal to the radius. The other forces acting on the drum, the normal from the axle and the weight, both act through the CM and thus do not create torque. The drum accelerates counterclockwise as the block moves down.



$\Sigma F_y = ma_y$	$\Sigma\tau\Sigma\tau_z = I_{zz}$
T - Mg = -Ma	RT = I

Since the rope is wrapped around the drum, we also have the relationship a = Rα .

Referring to the table of Moments of Inertia, we find that $I = ½mR^2$ for a solid cylinder. So our first equation is T = Mg - Ma. Our second is $RT = ½mR^2(a/R)$, or when we simplify T = ½ma. Putting this result into the first equation yields a = Mg / [M + ½m] = 1.353 m/s^2. Thus α = a/R = 2.706 rad/s^2. As well, T = ½ma = ½mMg / [M + ½m] = 84.56 N.

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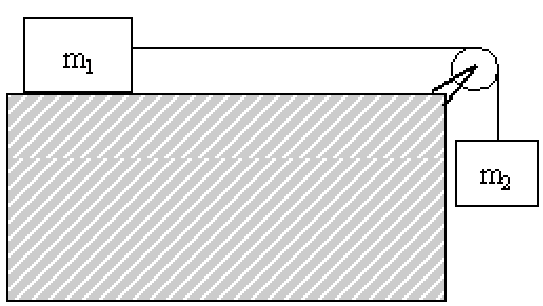
4. Two blocks are connected over a pulley as shown below. The pulley has mass M and radius R. What is the acceleration of the blocks and the tension in the rope on either side of the pulley? (HINT: The tension must be different or the pulley would not rotate.)



Since the problem wants accelerations and forces, and one object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for each object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the pulley has a fixed axle we need only consider the torques acting on it. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the block on the table are weight, a normal from the table, and tension. This block will accelerate to the right. The forces acting directing on the hanging block on the table are weight and tension. This block will accelerate downwards. The only forces directly acting on the pulley which creates torque are the tensions. Note that ropes, and therefore tensions, are always tangential to the object and thus normal to the radius. The tensions are different on either side of the pulley because of static friction - which we don't need to consider. The other forces acting on the pulley, the normal from the axle and the weight, both act through the CM and thus do not create torque. The pulley accelerates clockwise as the hanging block moves down.

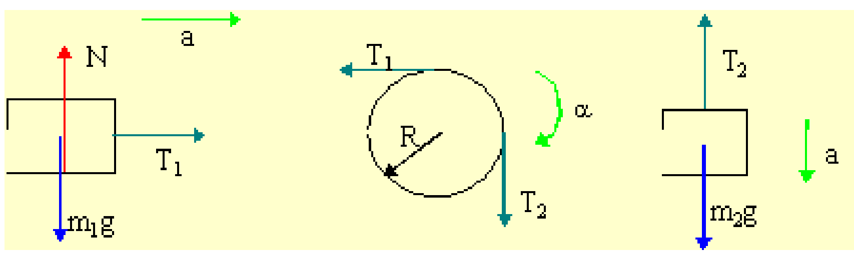


	Table Block	***Pulley***	***Hanging Block***
$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$	$\Sigma F_y = ma_y$
$T_1 = m_1 a$	$N - m_1 g = 0$	$RT_1 - RT_2 = -I$	$T_2 - m_2 g = -m_2 a$

From our table of Moments of Inertia, we find $I = ½MR^2$ for a solid disk. As well, since the rope is strung over the pulley, we know $a = R\alpha$. Using these facts, the third equation becomes $T_1 - T_2 = ½Ma$. Using the first equation, $T_1 = m_1 a$, and the second equation, $T_2 = m_2 g - m_2 a$, we can eliminate T_1 and T_2 from the third equation:

$$T_1 - T_2 = [m_1 a] - [m_2 g - m_2 a] = ½Ma .$$

Collecting terms that contain a, and rearranging yields,

$$a = m_2 g / [m_1 + m_2 + ½M].$$

The angular acceleration of the pulley is thus

$$\alpha = a/R = m_2 g / [m_1 + m_2 + ½M]R .$$

The tension in the left side of the rope is given by

$$T_1 = m_1 a = m_1 m_2 g / [m_1 + m_2 + ½M] .$$

The tension in the hanging potion of the rope is

$$T_2 = m_2g - m_2a = m_2g\{1 - m_2 / [m_1 + m_2 + ½M]\}.$$

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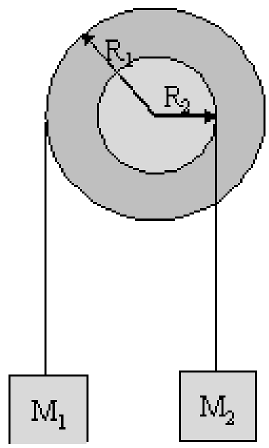
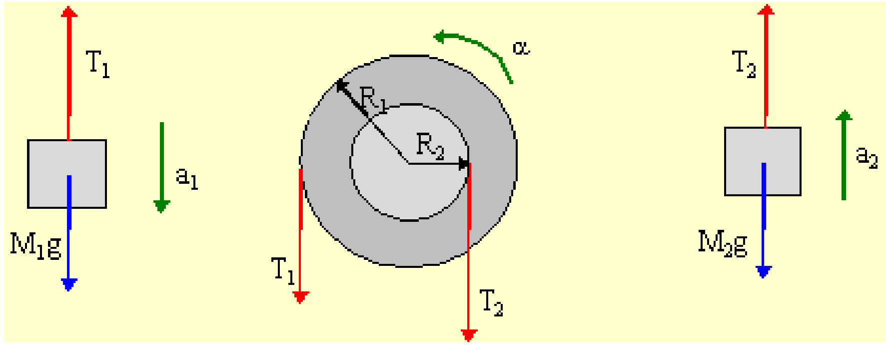
5. A winch has a moment of inertia of $I = 10.0$ kg-m^2. Two masses $M_1 = 4.00$ kg and $M_2 = 2.00$ kg are attached to strings which are wrapped around different parts of the winch which have radii $R_1 = 40.0$ cm and $R_2 = 25.0$ cm.
(a) How are the accelerations of the two masses and the pulley related?
(b) Determine the angular acceleration of the masses. Recall that each object needs a separate free body diagram.
(c) What are the tensions in the strings?



Since the problem wants accelerations and forces, and one object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for each object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the pulley has a fixed axle we need only consider the torques acting on it. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the hanging blocks are weight and tension. Let's assume M_1 accelerates downwards and thus M_2 upwards. The only forces directly acting on the pulley which create torque are the tensions. Note that ropes, and therefore tensions, are always tangential to the object and thus normal to the radius. The tensions are different on either side of the pulley since they are different ropes. The other forces acting on the pulley, the normal from the axle and the weight, both act through the CM and thus do not create torque. The pulley accelerates counterclockwise as a result of our assumption for the acceleration of the blocks.



Left	*Pulley*	*Right*
$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$	$\Sigma F_y = ma_y$
$T_1 - M_1g = -M_1a_1$	$R_1T_1 - R_2T_2 = I\alpha$	$T_2 - M_2g = M_2a_2$

(a) The acceleration of M_1 is equal to the tangential acceleration of the outside of the winch, so $a_1 = \alpha R_1$. The acceleration of M_2 is equal to the tangential acceleration of the inside ring of the winch, so $a_2 = \alpha R_2$.

(b) If we use the relationships from part (a), we can rewrite the equations in the table as

$$T_1 = M_1g - M_1R_1\alpha, \text{ and}$$

$$T_2 = M_2g + M_2R_2\alpha.$$

We use these results to eliminate T_1 and T_2 from the torque equation

$$R_1T_1 - R_2T_2 = I\alpha$$

$$R_1[M_1g - M_1R_1\alpha] - R_2[M_2g + M_2R_2\alpha] = I\alpha$$

$$R_1M_1g - R_2M_2g = [I + M_1(R_1)^2 + M_2(R_2)^2]\ \alpha.$$

Thus we find

$$\alpha = g(R_1M_1 - R_2M_2)/[I + M_1(R_1)^2 + M_2(R_2)^2] = 1.002 \text{ rad/s}^2 .$$

(c) Using this results, and our previous equations for the tension in each string, we find

$$T_1 = M_1g - M_1R_1\alpha = 38.60 \text{ N, and}$$

$$T_2 = M_2g + M_2R_2\alpha = 20.12 \text{ N} .$$

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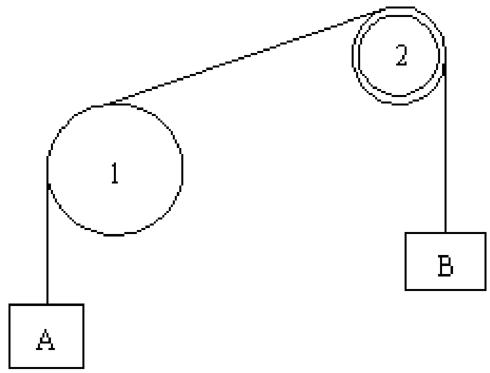
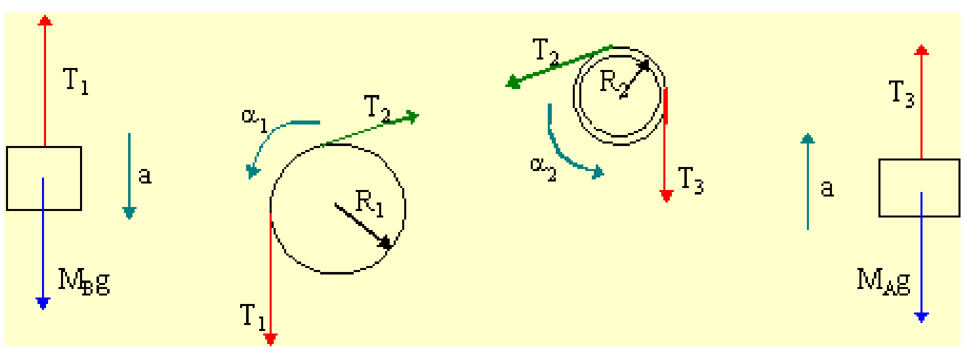
6. A rope connecting two blocks is strung over two real pulleys as shown in the diagram below. Determine the acceleration of the blocks and angular acceleration of the two pulleys. Block A is has mass of 10.0 kg. Block B has a mass of 6.00 kg. Pulley 1 is a solid disk, has a mass of 0.55 kg, and a radius of 0.12 m.

Pulley 2 is a ring, has mass 0.28 kg, and a radius of 0.08 m. The rope does not slip.



Since the problem wants accelerations and forces, and two objects rotate, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for each object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the pulleys have fixed axles, we need only consider the torques acting on each. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the hanging blocks are weight and tension. Let's assume A accelerates downwards and thus B upwards. The only forces directly acting on the pulleys which create torque are the tensions. Note that ropes, and therefore tensions, are always tangential to the object and thus normal to the radius. The tensions are different on either side of the pulley due to static friction with the surface of the pulleys. The other forces acting on each pulley, the normal from the axle and the weight, both act through the CM and thus do not create torque. The blocks are connected by the same rope and thus have the same magnitude of acceleration. The rope does not slip as it goes over the pulleys, so the pulleys have the same tangential acceleration as the rope. Since the pulleys have different radii, they have different angular accelerations.



Left	***Disk Pulley***	***Hoop Pulley***	***Right***
$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$	$\Sigma\tau_z = I_z\alpha_z$	$\Sigma F_y = ma_y$
$T_1 - M_A g = -M_A a$	$RT_1 - RT_2 = I_{disk}\alpha_1$	$R_2T_2 - R_2T_3 = I_{hoop}\alpha_2$	$T_3 - M_B g = M_B a$

In addition to the equations we have found above, we also know that the tangential acceleration of the pulleys is the same as the acceleration of the rope. Thus the angular acceleration of each pulley is related to a by $\alpha_1 = a/R_1$ and $\alpha_2 = a/R_2$. Examining a table of Moments of Inertia reveals that $I_{disk} = ½M_{disk}(R_1)^2$ and $I_{disk} = M_{hoop}(R_2)^2$. Using this information allows us to rewrite the equations as

$$T_1 = M_A g - M_A a \quad (1),$$

$$T_1 - T_2 = \tfrac{1}{2} M_{disk} a \quad (2),$$

$$T_2 - T_3 = M_{hoop} a \quad (3), \text{ and}$$

$$T_3 = M_B g + M_B a \quad (4).$$

If we add equations (2) and (3) together, we get

$$T_1 - T_3 = (\tfrac{1}{2} M_{disk} + M_{hoop}) a .$$

Then equations (1) and (4) can be used to eliminate T_1 and T_3 from the above

$$[M_A g - M_a a] - [M_B g + M_b a] = (\tfrac{1}{2} M_{disk} + M_{hoop}) a .$$

Collecting terms involving a and rearranging yields,

$$a = (M_A - M_B) g / (M_A + M_B + \tfrac{1}{2} M_{disk} + M_{hoop}) = 2.370 \text{ m/s}^2.$$

Using the above result, we find the angular accelerations

$$\alpha_1 = a/R_1 = 19.75 \text{ rad/s}^2$$

$$\text{and } \alpha_2 = a/R_2 = 29.63 \text{ rad/s}^2 .$$

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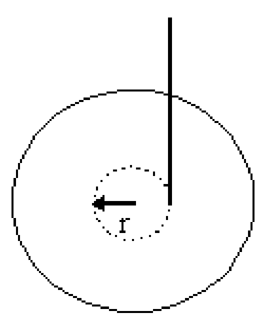
Rolling Objects

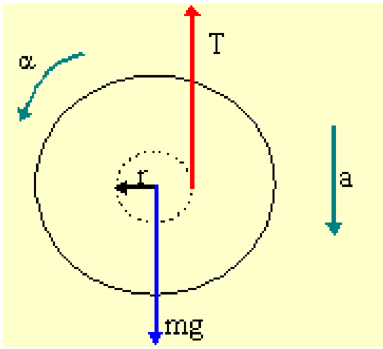
7. A yo-yo has a mass M, a moment of inertia I, and an inner radius r. A string is wrapped around the inner cylinder of the yo-yo. A person ties the string to his finger and releases the yo-yo. As the yo-yo falls, it does not slip on the string (i.e. the yo-yo rolls). Find the acceleration of the yo-yo.



Since the problem wants an acceleration, and an object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for each object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the yo-yo does not have a fixed axle, we need consider the torques acting about the CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the hanging yo-yo are weight and tension. Let's assume it accelerates downwards. The only force directly acting on the pulley which creates a torque is the tensions, the weight acts from the CM and cannot create a torque. Note that strings, and therefore tensions, are always tangential to the object and thus normal to the radius.

The yo-yo is said to roll without slipping. That phrase means that the angular acceleration of the yo-yo about its CM is related to its linear acceleration by $a = R\alpha$. Note that since the string is tied around the inner cylinder; it is that radius which figures into the relation.



$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$
$T - mg = -ma$	$rT = I\alpha_{cm}$

Since $\alpha_{cm} = a/r$, we have two simple equations

$$T = mg - ma, \text{ and } T = Ia / r^2.$$

Eliminating T from the first equation yields,

$$a = mg/[m+I/r^2] .$$

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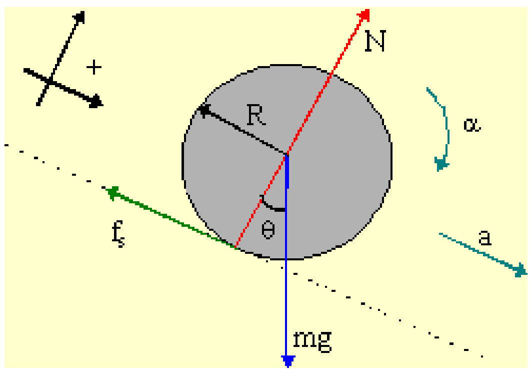
8. A solid cylinder rolls down an inclined plane without slipping. The incline makes an angle of 25.0 to the horizontal, the coefficient of static friction is $\mu_s = 0.40$, and $I_{cyl} = ½MR^2$. Hint - you may not assume that static friction is at its maximum!
(a) Find its acceleration.
(b) Find the angle at which static friction is at its maximum, at just above this angle the object will start to slip.

Since the problem wants an acceleration, and an object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for the object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the ball does not have a fixed axle, we need consider the torques acting about the CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directly on the ball are the normal, weight, and friction. Naturally the cylinder will accelerate downward the incline. Since the cylinder isn't slipping, its forward rate of rotation, its angular acceleration, is also forward. The only force directly acting on the cylinder which creates a torque is the friction, the normal and the weight act through the CM and cannot create a torque. Note that friction, being along the surface, is tangential to the cylinder and thus normal to the radius.

The cylinder is said to roll without slipping. That phrase means that the angular acceleration of the ball about its CM is related to its linear acceleration by $a = R\alpha$.

The type of friction is static since we are told that the cylinder is rolling without slipping. The only point left to resolve is in which direction it points. Since friction creates the only torque, and we have decided that α is forward, then friction must be up the incline. Only point to be careful about is that in rolling problems, one seldom is dealing with the $f_{s\ MAX}$ unless it is explicitly stated.



$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$
$mg\sin\theta - f_s = ma$	$N - mg\cos\theta = 0$	$-Rf_s = -I_{cyl}\alpha_{cm}$

(a) The first equation is $mg\sin\theta - ma = f_s$. The third equation can be simplified by using the given value for I_{cyl} and by noting that $\alpha_{cm} = a/R$. Then the third equation becomes

$$f_s = ½ma .$$

This can be substituted into the first equation to get

$$mg\sin\theta - ma = ½ma .$$

Solving for a yields

$$a = (2/3)g\sin\theta = 2.764 \text{ m/s}^2.$$

(b) We can use this result with $f_s = ½ma$, to get an expression for f_s,

$$f_s = (1/3)mg\sin\theta . \qquad (1)$$

However, the maximum value of f_s is $\mu_s N$. The second equation gives $N = mg\cos\theta$, so

$$f_s = \mu_s mg\cos\theta \, . \qquad (2)$$

Using (1) and (2) to eliminate f_s, yields

$$(1/3)mg\sin\theta = \mu_s mg\cos\theta \, .$$

Using the identity $\tan\theta = \sin\theta/\cos\theta$, we get

$$\theta = \tan^{-1}(3\mu_s) = 50.2^o \, .$$

This is the angle at which the cylinder would start to slip as it moved down the incline.

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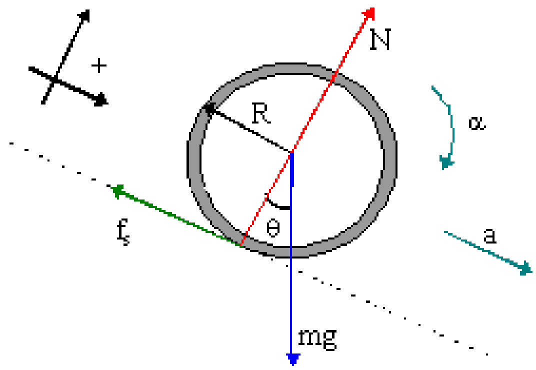
9. A thin-shelled cylinder rolls up an inclined plane without slipping. The incline makes an angle of 25.0 to the horizontal, the coefficient of static friction is $\mu_s = 0.40$, and $I_{hoop} = MR^2$.
(a) Find its acceleration.
(b) Find the angle which the object will start to slip.

Since the problem wants an acceleration, and an object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for the object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the hoop does not have a fixed axle, we need to consider the torques acting about the CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the hoop are the normal, weight, and friction. As the hoop goes up the incline it is slowing down, so its acceleration is down the incline. Since the hoop isn't slipping, its forward rate of rotation, its angular acceleration, is decreasing. The only force directly acting on the hoop which creates a torque is the friction, the normal and the weight act through the CM and cannot create a torque. Note that friction, being along the surface, is tangential to the hoop and thus normal to the radius.

The hoop is said to roll without slipping. That phrase means that the angular acceleration of the hoop about its CM is related to its linear acceleration by $a = R\alpha$.

The type of friction is static since we are told that the hoop is rolling without slipping. The only point left to resolve is in which direction it points. Since friction creates the only torque, and we have decided that α is backward, then friction must be up the incline. One point to be careful about is that in rolling problems, one seldom is dealing with the $f_{s\ MAX}$ unless it is explicitly stated.



$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma \tau_z = I_z\alpha_z$
$mg\sin\theta - f_s = ma$	$N - mg\cos\theta = 0$	$-Rf_s = -I_{hoop}\alpha_{cm}$

(a) The first equation is $mg\sin\theta - ma = f_s$. The third equation can be simplified by using the given value for I_{hoop} and by noting that $\alpha_{cm} = a/R$. Then the third equation becomes

$$f_s = ma \ .$$

This can be substituted into the first equation to get

$$mg\sin\theta - ma = ma \ .$$

Solving for a yields

$$a = \tfrac{1}{2}g\sin\theta = 4.146 \text{ m/s}^2 \ .$$

(b) We can use this result with $f_s = ma$, to get an expression for f_s,

$$f_s = \tfrac{1}{2}mg\sin\theta \ . \qquad (1)$$

However, the maximum value of f_s is $\mu_s N$. The second equation gives $N = mg\cos\theta$, so

$$f_s = \mu_s mg\cos\theta \ . \qquad (2)$$

Using (1) and (2) to eliminate f_s, yields

$$\tfrac{1}{2}mg\sin\theta = \mu_s mg\cos\theta \ .$$

Using the identity $\tan\theta = \sin\theta/\cos\theta$, we get

$$\theta = \tan^{-1}(2\mu) = 38.7^\circ \ .$$

This is the angle at which the hoop would start to slip as it moved down the incline.

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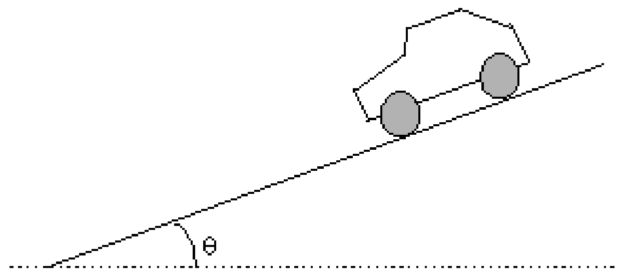
10. A toy car has a frame of mass M and four wheels of mass m. The wheels are solid disks. The car is placed on an incline and let go. Assume each tire supports one-quarter of the car's weight.
(a) Find the acceleration of the toy car.
(b) If the coefficient of static friction is μ, find an expression for the angle at which the wheels begin to slip.

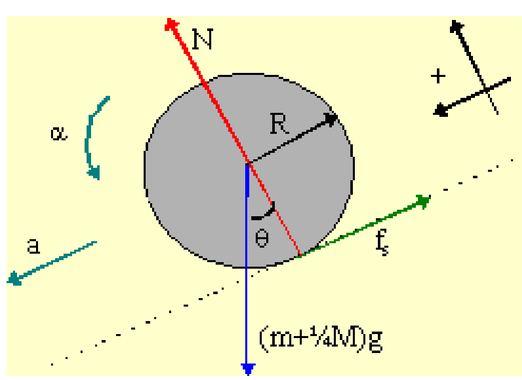


Since the problem wants an acceleration, and objects rotate, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for the object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the wheel axle is not fixed, we need consider the torques acting about each wheel's CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directly on each wheel are the normal, weight, and friction. Naturally the car will accelerate downward the incline. We will assume that each wheel supports one quarter of the car's weight and thus are all identical - we need only one FBD. Since the wheel isn't slipping, its forward rate of rotation, its angular acceleration, is also forward. The only force directly acting on the wheel which creates a torque is the friction, the normal and the weight act through the CM and cannot create a torque. Note that friction, being along the surface, is tangential to the wheel and thus normal to the radius.

The wheel is said to roll without slipping. That phrase means that the angular acceleration of the wheel about its CM is related to its linear acceleration by $a = R\alpha$.

The type of friction is static since we are told that the wheel is rolling without slipping. The only point left to resolve is in which direction it points. Since friction creates the only torque, and we have decided that α is forward, then friction must be up the incline. One point to be careful about is that in rolling problems, one seldom is dealing with the $f_{s\,MAX}$ unless it is explicitly stated.



$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$
$(m+\frac{1}{4}M)g\sin\theta - f_s = (m+\frac{1}{4}M)a$	$N - (m+\frac{1}{4}M)g\cos\theta = 0$	$Rf_s = I_{disk}\alpha_{cm}$

(a) The first equation is $mg\sin\theta - ma = f_s$. The third equation can be simplified by using $I_{hoop} = ½mR^2$ and by noting that $\alpha_{cm} = a/R$. Then the third equation becomes

$$f_s = ½ma\ .$$

This can be substituted into the first equation to get

$$(m+¼M)g\sin\theta - ½ma = (m+¼M)a\ .$$

Solving for a yields

$$a = g\sin\theta[(M+4m)/(M+6m)]\ .$$

(b) We can use this result with $f_s = ½ma$, to get an expression for f_s,

$$f_s = ½mg\sin\theta[(M+4m)/(M+6m)]\ . \quad (1)$$

However, the maximum value of f_s is $\mu_s N$. The second equation gives $N = (m+¼M)g\cos\theta$, so

$$f_s = \mu_s(m+¼M)g\cos\theta\ . \qquad (2)$$

Using (1) and (2) to eliminate f_s, yields

$$½mg\sin\theta[(M+4m)/(M+6m)] = \mu_s(m+¼M)g\cos\theta\ .$$

Using the identity $\tan\theta = \sin\theta/\cos\theta$, we get

$$\theta = \tan^{-1}([M+6m]/2m)\ .$$

This is the angle at which the wheels would start to slip as it moved down the incline.

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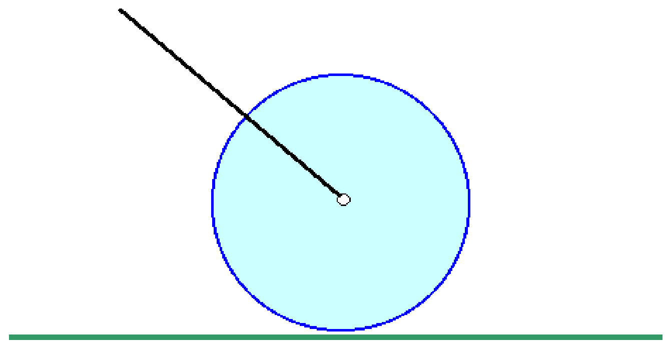
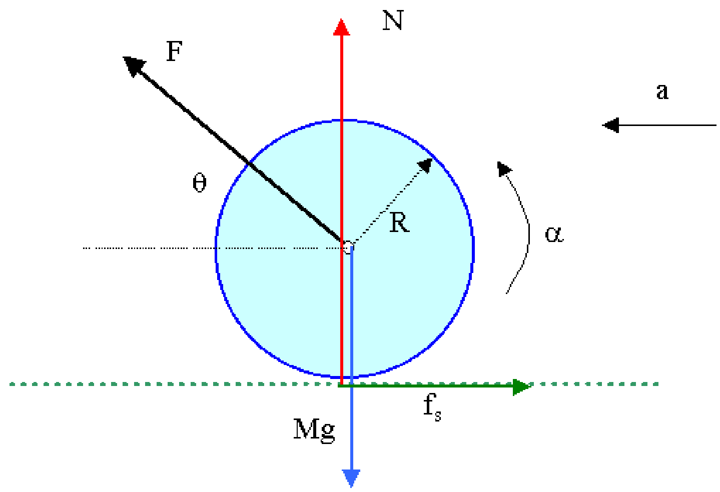
11. A person pulls a heavy lawn roller by the handle with force F so that it rolls without slipping. The handle is attached to the axle of the solid cylindrical roller. The handle makes an angle θ to the horizontal. The roller has a mass of M and a radius R. The coefficients of friction between the roller and the ground are μ_s and μ_k.
(a) Find the acceleration of the roller.
(b) Find the frictional force acting on the roller.
(c) If the person pulls too hard, the roller will slip. Find the value of F at which this occurs.



First we draw the FBD. Clearly we have F, weight, and a normal force acting on the roller. As well, there must be static friction since we have rolling without slipping. It is not at a maximum since the roller only starts to slip in part (c). The direction of f_s must be to the right, since the force F pulls the roller into the ground, the ground pushes back. We assume the accelerations are as shown, left and ccw. Note that these are consistent.



We apply Newton's Second Law to the problem.

$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma \tau_{cm} = I_{cm}\alpha_{cm}$
$-F\cos(\theta) + f_s = -Ma$	$N + F\sin(\theta) - Mg = 0$	$Rf_s = I\alpha$

We also know that $a = R\alpha$ and $I = ½MR^2$. We substitute these relationships into the torque equation

$$Rf_s = ½MR^2(a/R) \tag{1}$$

This yields an equation for f_s,

$$f_s = ½Ma \tag{2}$$

We take this result and substitute it into the x-component equation

$$-F\cos(\theta) + ½Ma = -Ma \tag{3}$$

Solving for a, as required in part (a), yields

$$a = (2/3)(F/M)\,cos(\theta) \quad (4)$$

We can also find f_s, as required in part (b), by substituting (4) back into Eqn. (2);

$$f_s = (1/3)F\,cos(\theta) \quad (5)$$

Part (c) asks us when the roller will slip. This occurs when $f_s = f_s^{max}$. Now we know $f_s^{max} = \mu_s N$. We find N from the y-component equation, so

$$f_s^{max} = \mu_s[Mg - F\,sin(\theta)] \quad (6)$$

Equating Eqns. (5) and (6) will tell us the value of F at which slipping occurs

$$(1/3)F\,cos(\theta) = \mu_s[Mg - F\,sin(\theta)] \quad (7)$$

We get F by itself on the left-hand side

$$F[(1/3)cos(\theta) + \mu_s sin(\theta)] = \mu_s Mg \quad (8)$$

So

$$F = \frac{3\mu_s Mg}{\cos(\theta) + 3\mu_s \sin(\theta)}. \quad (9)$$

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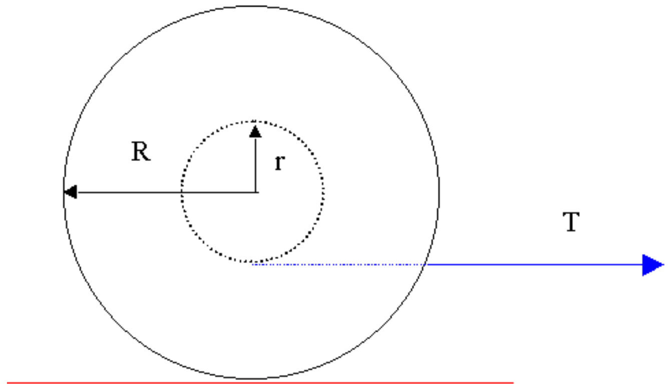
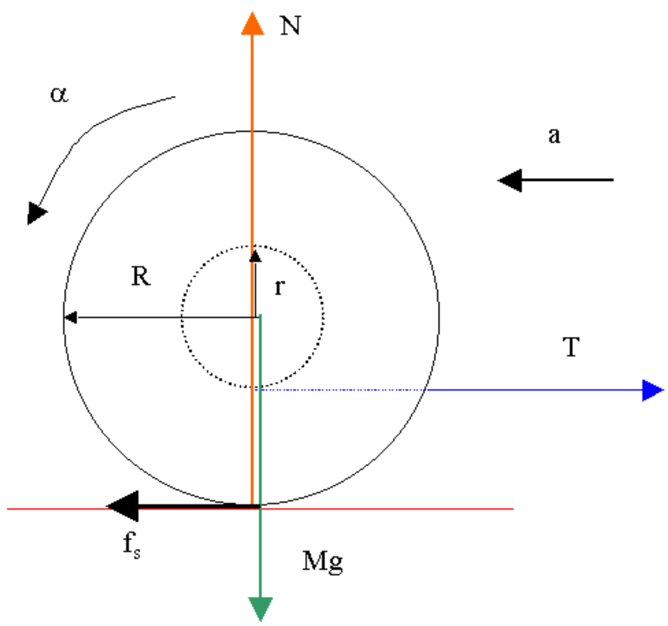
12. A yo-yo of Mass M, moment of inertia I, and inner and outer radii r and R, is gently pulled by a string with tension T as shown in the diagram below. The coefficients of friction between the yo-yo and the table are μ_s and μ_k.
(a) Find the acceleration.
(b) Find the friction acting on the yo-yo.
(c) At what value of T will the yo-yo begin to slip?



We know there is a normal and weight acting on the yo-yo but these do not create torques as they operate on or through the Centre of Mass. There is non-maximum static friction acting but we must determine it's direction. First if there were no T, there would be no friction. T is twisting the yo-yo counterclockwise pushing the yo-yo into the surface. The surface reacts by pushing back. The FBD looks like



Here I have guessed that the yo-yo will roll backwards. Notice that my choice of a and α are consistent.

I apply Newton's Second Law

$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma \tau_{cm} = I_{cm}a_{cm}$
$T - f_s = -Ma$	$N - Mg = 0$	$rT - Rf_s = I\alpha$

We also know $a = R\alpha$.

The torque equation becomes can be solved for f_s

$$f_s = (r/R)T - (I/R^2)a \ . \quad (1)$$

We substitute this into the x-component equation

$$T - [(r/R)T - (I/R^2)a] = - Ma \tag{2}$$

We bring the term involving a from the left to the right and solve for a in terms of T,

$$T(R\text{-}r)/R = - (M + I/R^2)a \tag{3}$$

or

$$a = -\frac{(R - r)R}{MR^2 + I} T . \tag{4}$$

The fact that a is negative tells me that my guess about the direction of a and α are wrong. The acceleration is forward and counterclockwise.

Substituting Eqn. (4) into Eqn. (1), we find

$$f_s = \frac{MRr + I}{MR^2 + I} T . \tag{5}$$

Since f_s is positive, it must be have been chosen in the right direction.

Now as we see from Eqn. (5), as T increases so does f_s. We know f_s £ $\mu_s N$ or f_s £ $\mu_s Mg$ in our case when we make use of the y-component equation. Thus we have a limit on T

$$\frac{MRr + I}{MR^2 + I} T \le \mu_s Mg . \tag{6}$$

This yields the result

$$T \le \mu_s Mg \frac{MR^2 + I}{MRr + I} . \tag{7}$$

If T is any bigger, the yo-yo will slip.

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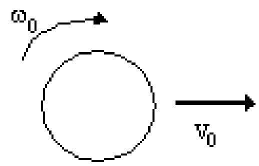
Slipping Objects

13. A bowling ball of Radius R is given an initial velocity of v_0 down the lane and a forward spin of $w_0 = 3v_0/R$. It first slips when it makes contact with the lane, but will eventually start to roll without slipping. The coefficient of kinetic friction is μ_k.
(a) What is the direction of the frictional force? Explain.

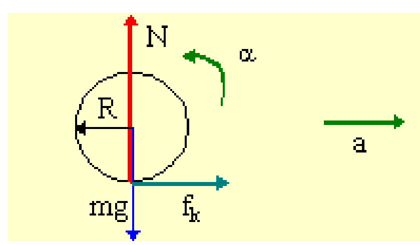
(b) For how long does the ball slide before it begins to roll without slipping?
(c) What is the speed of the bowling ball when it begins to roll without slipping?
(d) What distance does the ball slide down the lane before it starts rolling without slipping?



Since the problem involves forces, and an object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for the object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the bowling ball does not have a fixed axle, we need to consider the torques acting about the CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the bowling ball are the normal, weight, and friction. The only force directly acting on the bowling ball which creates a torque is the friction, the normal and the weight act through the CM and cannot create a torque. Note that friction, being along the surface, is tangential to the bowling ball and thus normal to the radius.

The bowling ball is said to be slipping. That phrase means that the angular acceleration of the bowling ball about its CM is NOT related to its linear acceleration, i.e. by a ¹ rα. It also means that we are dealing with kinetic friction. The only point left to resolve is in which direction it points. To do this examine the tangential velocity of the outside rim of the bowling ball, $v = R\Omega_0 = 3v_0$. This means the rim is spinning much faster than the CM is moving forward; friction will act to slow the rim as the rim rubs on the surface of the bowling lane. Friction will point in the direction of the initial velocity. Since friction causes the only torque, the angular acceleration is backwards.



$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$
$f_k = ma$	$N - mg = 0$	$Rf_k = I_{cm}\alpha$

We also know that $f_k = \mu N$. Since the second equation gives $N = mg$, we have $f_k = \mu mg$. The first equation thus yields

$$a = \mu g. \qquad (1)$$

A bowling ball is a solid sphere and using a table of moments of inertia we find $I = (2/5)mR^2$. With our expression for f_k, our third equation becomes

$$R\mu mg = (2/5)mR^2\alpha_{cm}\,,$$

or on rearranging,

$$\alpha_{cm} = (5/2)\mu g/R\,. \qquad (2)$$

Time is a kinematics variable. We have initial velocities and accelerations so we can write expressions for the linear and rotational velocity as a function of time

$$v(t) = v_0 + at = v_0 + \mu gt\,, \qquad (3)$$

and

$$\Omega(t) = \Omega_0 - \alpha t = [3v_0 - (5/2)\mu gt]/R\,. \qquad (4)$$

We want the time when

$$v(t) = R\Omega(t)\,.$$

Substituting in equation (3) and (4), we get

$$v_0 + \mu gt = 3v_0 - (5/2)\mu gt\,.$$

Solving for t yields

$$t = 4v_0 / 7\mu g\,.$$

This is the time that is takes for the bowling ball to start to roll without slipping. Plugging this result back into equation (3) gives us the linear velocity of the bowling ball at this and all later times

$$v(t) = v_0 + g[4v_0 / 7\mu g] = (11/7)v_0.$$

Making use of another kinematics formula, we find the distance traveled,

$$\Delta x = v_0t + ½at^2 = v_0[4v_0/7\mu g] + ½[\mu g][4v_0/7\mu g]^{\,2} = 36(v_0)^2 / 49\mu g\,.$$

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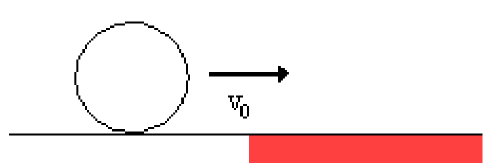
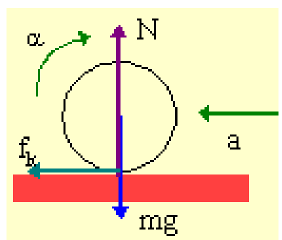
14. A solid sphere is sliding (not rolling!) across a frictionless surface with speed v_0. It slides onto a surface where the coefficient of kinetic friction is μ. Eventually it will start to roll without slipping.
(a) What is the direction of the frictional force? Explain.
(b) For how long does the sphere slide before it begins to roll without slipping?
(c) What is the speed of the sphere when it begins to roll without slipping?
(d) What distance does the sphere slide it starts rolling without slipping?



Since the problem involves forces, and an object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for the object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the ball does not have a fixed axle, we need to consider the torques acting about the CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma \tau_z = I_z \alpha_z$ to get a set of equations.

The forces acting directing on the ball are the normal, weight, and friction. The only force directly acting on the ball which creates a torque is the friction, the normal and the weight act through the CM and cannot create a torque. Note that friction, being along the surface, is tangential to the ball and thus normal to the radius.

The ball is said to be slipping. That phrase means that the angular acceleration of the bowling ball about its CM is NOT related to its linear acceleration, i.e. by a ¹ rα. It also means that we are dealing with kinetic friction. The only point left to resolve is in which direction it points. To do this examine the tangential velocity of the outside rim of the bowling ball, v = 0 since it wasn't rotating. This means the rim will rub on the rough surface as it moves to the right and thus it will experience kinetic friction to the left opposite to the direction of the initial velocity. Since friction causes the only torque, the angular acceleration is forward.



$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma \tau_z = I_z \alpha_z$
$-f_k = -ma$	$N - mg = 0$	$-Rf_k = -I\alpha_{cm}$

We also know that $f_k = \mu N$. Since the second equation gives N = mg, we have $f_k = \mu mg$. The first equation thus yields

$$a = \mu g. \qquad (1)$$

Consulting a table of moments of inertia, we find $I = (2/5)mR^2$ for solid sphere. With our expression for f_k, our third equation becomes

$$R\mu mg = (2/5)mR^2\alpha_{cm},$$

or, on rearranging,

$$\alpha_{cm} = (5/2)\mu g/R. \qquad (2)$$

Our results indicate that the forward motion slows as the forward rotation increases.

Time is a kinematics variable. We have initial velocities and accelerations so we can write expressions for the linear and rotational velocity as a function of time

$$v(t) = v_0 + at = v_0 - \mu gt , \qquad (3)$$

and

$$\Omega(t) = \Omega_0 + \alpha t = -(5/2)\mu gt/R . \qquad (4)$$

We want the time when $|v(t)| = |R\Omega(t)|$, where the absolute bars are there to stress that we are relating magnitudes and must be careful with signs.

Substituting in equation (3) and (4), we get

$$|v_0 - \mu gt| = |-(5/2)\mu gt| .$$

Solving for t yields

$$t = 2v_0 / 7\mu g .$$

This is the time that is takes for the bowling ball to start to roll without slipping. Plugging this result back into equation (3) gives us the linear velocity of the bowling ball at this and all later times

$$v(t) = v_0 - \mu g[2v_0 / 7\mu g] = (5/7)v_0 .$$

Making use of another kinematics formula, we find the distance traveled,

$$\Delta x = v_0 t + ½at^2 = v_0[2v_0/7\mu g] + ½[-\mu g][2v_0/7\mu g]^2 = 12(v_0)^2 / 49\mu g .$$

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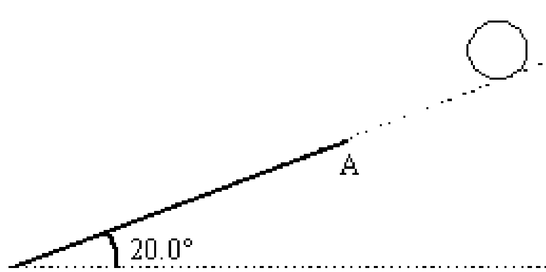
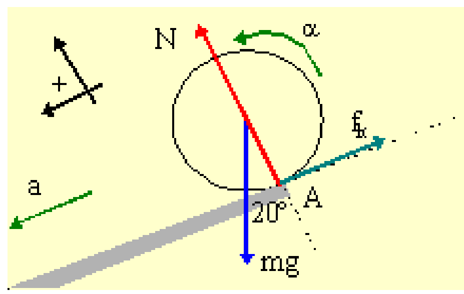
15. A ball is placed on an incline as shown in the diagram below. The upper part of the incline is frictionless, so the ball slides but does NOT rotate. At point A, when its speed is 4.50 m/s, it reaches a rough portion of the incline where $\mu_k = 0.20$. Here the ball starts to slip.
(a) How long does it take for the ball to roll without slipping?
(b) How far down the incline from point A does this occur?
(c) What is the speed of the ball when it starts to roll without slipping?



Since the problem involves forces, and an object rotates, that suggests we must use both the linear and rotational versions of Newton's Second Law. Applying Newton's Second Law requires that we draw free body diagrams for the object. In particular for any rotating body we must draw an extended FBD in order to calculate the torques. Since the ball does not have a fixed axle, we need to consider the torques acting about the CM. Once the diagrams are drawn, we use $\Sigma F_x = ma_x$, $\Sigma F_y = ma_y$, and $\Sigma\tau_z = I_z\alpha_z$ to get a set of equations.

The forces acting directing on the ball at point A are the normal, weight, and friction. The only force directly acting on the ball which creates a torque is the friction, the normal and the weight act through the CM and cannot create a torque. Note that friction, being along the surface, is tangential to the ball and thus normal to the radius.

The ball starts to slip at A. That phrase means that the angular acceleration of the bowling ball about its CM is NOT related to its linear acceleration, i.e. by a ¹ rα. It also means that we are dealing with kinetic friction. The only point left to resolve is in which direction it points. To do this examine the tangential velocity of the outside rim of the bowling ball; v = 0 since it wasn't rotating. This means the rim will rub on the rough surface as it moves down the incline and thus it will experience kinetic friction up the incline opposite to the direction of the velocity. Since friction causes the only torque, the angular acceleration is counterclockwise.



$\Sigma F_x = ma_x$	$\Sigma F_y = ma_y$	$\Sigma\tau_z = I_z\alpha_z$
$-f_k + mg\sin\theta = ma$	$N - mg\cos\theta = 0$	$Rf_k = I\alpha_{cm}$

We also know that $f_k = \mu N$. Since the second equation gives $N = mg\cos\theta$, we have $f_k = \mu mg\cos\theta$. The first equation thus yields

$$a = g(\sin\theta - \mu\cos\theta) . \qquad (1)$$

Consulting a table of moments of inertia, we find $I = (2/3)mR^2$ for a hollow sphere. With our expression for f_k, our third equation becomes

$$R\mu mg\cos\theta = (2/3)mR^2\alpha_{cm} ,$$

or, on rearranging,

$$\alpha_{cm} = (3/2)\mu g\cos\theta/R . \qquad (2)$$

Our results indicate that the linear motion is slower than with no friction and that the counterclockwise rotation increases.

Time is a kinematics variable. We have initial velocities and accelerations so we can write expressions for the linear and rotational velocity as a function of time

$$v(t) = v_0 + at = v_0 + g(\sin\theta - \mu\cos\theta)t, \qquad (3)$$

and

$$\Omega(t) = \Omega_0 + \alpha t = (3/2)\mu g\cos\theta t/R . \qquad (4)$$

We want the time when v(t) = RΩ(t), where the absolute bars are there to stress that we are relating magnitudes and must be careful with signs.

Substituting in equation (3) and (4), we get

$$v_0 + g(\sin\theta - \mu\cos\theta)t = (3/2)\mu g\cos\theta t .$$

Collecting the terms involving t together

$$v_0 = g[(3/2)\mu\cos\theta - (\sin\theta - \mu\cos\theta)]t ,$$

or, more simply

$$v_0 = g[(5/2)\mu\cos\theta - \sin\theta]t ,$$

Solving for t yields

$$t = v_0 / g[(5/2)\mu\cos - \sin\theta] .$$

This is the time that is takes for the bowling ball to start to roll without slipping. Notice that the denominator could be zero or even negative for some angles - that is at those angles the ball never stops slipping. Plugging the given values into this result, we get t = 3.5886 seconds.

Putting this result back into equation (3) gives us the linear velocity of the bowling ball at this and all later times

$$v(t) = v_0 + g(\sin\theta - \mu\cos\theta)\ \{v_0/g[(5/2)\mu\cos\theta - \sin\theta]\} = v_0\ \{3\mu\cos\theta / [5\mu\cos\theta - 2\sin\theta]\} .$$

With the given values this becomes v_f = 9.9243 m/s.

Making use of another kinematics formula, we find the distance traveled,

$$\Delta x = ½(v_0 + v_f)t = ½v_0\{1 + 3\mu\cos\theta / [5\mu\cos\theta - 2\sin\theta]\}\ \{v_0/ g[(5/2)\mu\cos\theta - \sin\theta]\}$$

or, after rearranging,

$$\Delta x = [(v_0)^2/g]\ [8\mu\cos\theta - 2\sin\theta] / [5\mu\cos\theta - 2\sin\theta]^2 .$$

The numeric value of which is x = 25.88 m .

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