Building Conceptual Understanding in Precalculus

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Abstract:

The authors discuss their attempts to produce "conceptual understanding" in a university-level precalculus course and the challenges it presented. The authors first discuss the importance of building understanding of math concepts and moving students away from memorization of formulas and provide examples of the types of problems students are asked to confront. Results of surveys show that students are initially uncomfortable with this approach as it violates their expectations about "what math is." The authors discuss their challenges with finding ways to assess this form of complex reasoning and how best to partner with students to enhance their learning.

Key Words:

Precalculus, conceptual understanding, math concepts, complex reasoning.

Introduction

Picking up a random student transcript these days, you will often see that the student has completed a math course by the name of Precalculus. However, on the vast majority of those same transcripts (85-90% at Rockhurst), you will not see a Calculus course listed anywhere. It naturally compels us to ask the question, "Why are students taking a Precalculus course if they are not continuing on to Calculus?". Although the "Pre-calculus" course was originally developed to prepare college students for Calculus, the truth is that today most Precalculus students nationwide do not continue on to complete a calculus course (even of those college students nationwide who *intend* on completing both precalculus and calculus, only 30-40% successfully complete the two course sequence). At the same time, the number of students taking Precalculus has increased and diversified—both at Rockhurst and nationwide. So, who is taking precalculus and why are they taking it?

Currently, about 65% of Rockhurst students satisfy their math proficiency requirement (one college level math course required by our core) by taking Precalculus, and these students range from Business majors to Nursing majors with any number of Biology majors, Theology majors, Psychology majors, and a variety of others in between. In fact, several of these disciplines at Rockhurst (business and nursing, for instance), specifically require that their students satisfy the math proficiency requirement by taking Precalculus.

If these programs are not requiring or encouraging their students to enroll in this course to prepare for Calculus, what do they see in Precalculus? One of the major reasons for the popularity of this course among these diverse disciplines at Rockhurst is the work that our math department has done over the years in building a vision for the Precalculus course that supports the aims and objectives of other disciplines while still developing the desired mathematical thinking of students in what is most likely their final college math course. No single subset of quantitative skills or methods would be applicable to a large number of disciplines at once, nor would such learning objectives serve any student for very long past the final exam in Precalculus. Our vision, instead, centers around two main beliefs: (1) the most relevant, useful and lasting outcome that students will take away from this course is achieving an understanding of the *meaning* of mathematical concepts (i.e., *conceptual understanding*), and (2) the concepts of this course should build and support the understanding of mathematical modeling of data.

A student who achieves conceptual understanding in our course is understanding the meaning of a concept well enough (or deeply enough) to be able to adapt, modify and expand that concept in order to apply it in novel situations or novel ways (within an application or within a purely mathematical context). Perry (1999) reported that Cornfield and Knefelkamp (1979) characterized conceptual understanding as the ability to adapt, modify, and expand on a concept. They argued that this level of understanding enables a student "to investigate and compare things and to make judgements about adequacy or inadequacy, appropriateness or inappropriateness." (p.1) This view of conceptual understanding is synonymous with deep learning, which has acknowledged links to positive college student outcomes (Laird, Shoup, Kuh, & Schwarz, 2008; Ramsden, 1992). For the study of mathematics to be more relevant and applicable for non-mathematics students conceptual understanding is essential. It is also extremely important for future mathematics majors. It is impossible to expose a student to all of the situations in which they may need to apply a given concept in their discipline, even if that discipline is mathematics. For this reason, it is important that a student truly understands each mathematical concept and is able to reason from that understanding to answer questions—as opposed to simply memorizing (or mimicking) a few methods utilizing that concept.

In society today, we have the ability to collect, store, and access more data than was even imaginable fifty years ago. Students will encounter data in almost any discipline they choose or have chosen to study. It is imperative that they are able to use and interpret data effectively while also critically analyzing the conclusions that others in their discipline have drawn from data. Mathematical modeling of data is a way to approach a quantitative question by examining appropriate observed data and drawing reasonable conclusions based upon mathematical analysis of that data. In our

Precalculus classroom, this mathematical analysis involves examining—both numerically and graphically—observed data that relates two varying numerical quantities (for example, years since 1970 and minimum ozone readings—as shown in boxed example below), recognizing any relationships that may exist between these two varying quantities (for example, as years since 1970 increases at a steady rate mimimum ozone readings decrease by less and less), finding a mathematical function that describes that specific relationship (specifically, a mathematical, symbolic description of how to relate two changing quantities), using that mathematical function to draw conclusions or make predictions where we do not have observed data (for example predicting the minimum ozone readings in the year 2020), and finally evaluating the reasonableness of those conclusions/predictions based upon the accuracy of the model. By its very nature, mathematical modeling requires students to use conceptual understanding to think critically about a wide variety of quantitative questions, as the questions that we ask and the data available to answer them are extensive, diverse and ever changing.

Modeling Ozone Data

Data obtained from NASA TOMS (Total Ozone Mapping Spectrometer) website at
http://toms.gsfc.nasa.gov/eptoms/dataqual/ozone_v8.html.
Measurements are taken from approximately Mid Sept to Mid Oct in Antartica.

Min Ozone Reading

Modeling Function:
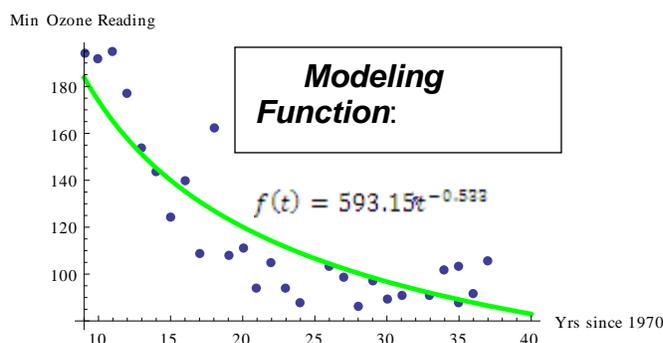
$$f(t) = 593.15t^{-0.533}$$

Use modeling function to make predictions:

- Predicted minimum ozone reading in year 2020:

$$= f(50) = 593.15(50)^{-0.533} = 73.7\ DU$$

- Predicted year when minimum ozone reading reaches a level of 50 DU:

$$f(t) = 593.15t^{0.533} = 50$$

$$t = 103.6\ years\ since\ 1970 = 2073.6$$

This article will focus on the challenge of providing the learning environment and student activities that help our students achieve conceptual understanding in Precalculus. We will describe why we believe that conceptual understanding can be built through an active-learning, student-centered Precalculus classroom where students are challenged to discover concepts and connect their discoveries with their own prior understanding and the world around them. We will also share our approach to assessing whether our students have achieved conceptual understanding and the results of our assessment efforts to date.

Building Conceptual Understanding

Most students enter our Precalculus class with the belief that "the method" is everything. It is always surprising to us when some students glance at a question and almost immediately pronounce that they "don't know how to do it." We know those students don't actually mean this because they could not possibly know this without

spending more than ten seconds thinking about the question. What those students mean is that they do not immediately know "the method." They are attempting to recognize and classify the question as a certain "type of problem" and then access the memorized sequence of steps ("the method") necessary to calculate the answer. If the question is not of a type that they recognize, they then conclude that they are unable to answer the question. This is worrying not only because those students do not understand how to read a question, think about it, discuss it and attempt to develop an approach to the question themselves building from the concepts involved, but more importantly because they don't even consider that process a possibility.

Our general sense that many students were coming to our course approaching problems in this very restricted way was confirmed when we reviewed the results of an online attitude survey that our students took during the first few days of the semester. One question asked students to "[i]ndicate how much you agree or disagree: In order to solve a challenging math problem, I NEED...:" Students chose a value on a scale of one through seven for each of seven options. The response "1" corresponded to "Not at all" and the response "7" corresponded to "Very Much." Over all seven sections of our Precalculus course (during the Fall 2009 semester) the highest scoring option was "To work hard" with an average response of 6.316 on the scale of 1 to 7. The next highest averages were the options "To have lots of practice in solving similar problems" and "To remember a lot of examples that I might use in constructing a solution" with overall average responses of 6.007 and 5.549, respectively. These two options—both of which indicate a perceived need to recognize and categorize a problem in order to solve it— were ranked significantly higher than the remaining options (including "To try multiple approaches to constructing a solution," "To use rigorous reasoning," and "To have freedom to do the problem in my own way"), which together averaged a response of only 4.5 over all sections.

Another question on this online attitude survey supported our belief that most students rely heavily on memorizing and using methods to solve problems. On this question, students scored "Learning specific procedures for solving math problems" and "Memorizing the sets of facts important for doing math" as "Very Important" with average responses of 5.43 and 5.50, respectively, on a scale of 1="Not at all Important" to 7 = "Extremely Important".

Upon reflection, these results should not be surprising to us, and are similar to attitudes found by Bookman & Friedman (2010). They found that a reformed calculus course "violated students' deeply held beliefs about the nature of mathematics and how it should be learned." In a typical, traditional mathematics classroom, the instructor begins by presenting definitions and explanations of new concepts to the class, perhaps also providing some level of rationalization or context by linking the new concepts to past and future classwork. The instructor then works some sample problems for the students, illustrating ways in which this new concept can be applied to answer questions or solve problems. During this process, students may be active in some way, but generally in this traditional classroom, it is the instructor alone that actively engages in the challenging task of reasoning from his/her own understanding of the meaning of a concept to the solution of a problem or the development of a new concept. The students, on the other hand, are passively watching most of this process. They may

understand the teacher's explanations and reasoning. They may even agree that the steps the teacher took build logically from the concept. They may even be actively working through portions of the problem (e.g., completing algebraic work, computations, etc.), but they never have to reason themselves from their own understanding of the concept to arrive at an approach and an answer to a given question. When these students are then faced with a question or problem to solve themselves, they are necessarily approaching it by considering the sample problems that they've seen, selecting one that is most like this new question/problem and using (perhaps even adapting slightly) "the method" of that sample problem to find the answer. In doing this, they are essentially mimicking their teacher's reasoning, which requires only minimal understanding of the concepts involved. Students come away from this type of learning able to solve a lot of useful problems, but their abilities are limited to the types of problems that they've already seen. Another concern, especially in a discipline that is widely accepted as building strong problem-solving and critical-thinking skills, is whether this type of learning truly requires critical thinking on the part of the students.

The effects of our students' previous mathematics learning experiences could again be seen when reviewing the results of the online attitude survey administered at the beginning of the semester. Students "strongly agreed" that they "learn mathematics best" when "[t]he instructor explains the solutions to problems," and when "[t]he homework assignments are similar to the examples considered in class." These two options scored an average response over all sections of precalculus in Fall 2009 of 6.373 and 6.379, respectively, on a scale of 1 to 7 (where "1" signified "Strongly Disagree" and "7" signified "Strongly Agree").

In our precalculus classroom, we strive to push our students beyond this traditional approach to learning mathematics so as to broaden their problem-solving abilities and to strengthen their critical-thinking skills. Specifically, we want them to be able to reason *independently* from a deep understanding of concepts (or prior knowledge) to answer questions and solve novel problems. Providing plenty of opportunities (some guided and some unguided) for students to engage in this type of reasoning in novel situations (where there is no example to work from) slowly builds the confidence and ability to reason in this way independently. It is this very experience, in turn, that also helps build further conceptual understanding.

From the very beginning of the course, students have to reason from concepts (and prior knowledge) to answer questions. Questions that we pose prompt students to discover the concepts of the course as opposed to being "presented" with the concepts through reading the text or listening to an instructor's explanations. Following these initial discovery questions, we pose additional "synthesis" questions that provide students with the opportunity to adapt, extend, and integrate their understanding of the concepts (in new settings), as this is an integral part of developing their ability to reason using those concepts. These synthesis questions may be embedded in an applied modeling investigation or they may be used to discover new concepts. In mathematics, as in any subject, a student is truly learning when they are asked to carefully examine their current understanding and then use it to develop new understanding.

An Example - The First Day of Class:

Our precalculus students walking into their first day of class are surprised. The very first class activity they are faced with is the following:

Problem 1:

Problem: During criminal investigations, footprints are often used to attain a description of the suspect. In particular, a person's footprint can be used to predict his/her height.

Develop a Process: Working together in small groups, design a process to predict, from a footprint, the height of the person who made that footprint. Discuss completely as a group and make sure that you all agree on the process. Then explain your process in detail below.

Make a Prediction: Using the process that you have developed, predict the height of the people who made the following footprints. Footprints (2-dimensional outlines of footprints – no depth) will be passed around the classroom.

The instructor hands out the activity and tells the students that there are measuring tapes and graph paper at the front of the room in case they need them for any reason during the activity. At first, students are reluctant to do anything with the activity. They each read it individually and look nervously at the person on either side of them. Finally, someone caves and discussion begins. However, after an initial few minutes of talking, some groups fall silent again. When the instructor approaches one such group and inquires what they are thinking about, they shrug and say they are not thinking because they do not know how to do the problem. The instructor encourages the group to try thinking about it again and asks if they knew that the footprint was bigger than their own footprint would that help them? What if it was smaller? This starts the group talking again and the instructor moves on. The classroom starts to look a little chaotic as a number of students have started to measure their own feet. Other students see them doing this and decide to give it a try themselves. One student is stretching her arms out as far as they can go on either side of her, and another student is holding her shoe up to her arm. When most groups seem to have come up with a process, the class comes together to discuss their different approaches. They debate how reasonable they think each approach is, compare and contrast the approaches, and in the end ensure that each group has an approach that they believe should give a good prediction. The following are two examples of approaches that students have come up with in our classrooms:

Approach 1	Approach 2
Model: Height = 6.5 x footprint length	Model: Height = 5.88 x footprint length
Student Explanation: The group explained: "Your wingspan is the same as your height." Then based upon approximate measurements from their own bodies, they concluded that: "Your foot is the same length as the distance from your wrist to your elbow which is roughly the same as the distance from your elbow to you shoulder. The distance between your shoulders is approximately one and a half times the length of your foot and your hands are each approximately half the length of your foot."	Student Explanation: "There were four people in our group - Tijona, John, Rachel and Justin. For each person we measured their footprint length and height. We found the value that we had to multiply the footprint length by in each case to get the person's height and then averaged these values (5.88) to find the general relationship."

Notice: Approach 2 may seem more mathematical (because they are doing some numerical calculations) but they are assuming proportionality (that we will be able to multiply the length of the footprint by a constant and get the height) without any justification. Whereas Approach 1 comes up with their proportionality statement from a physical process which inherently provides a justification for that statement and therefore makes it more mathematically correct. Justifying/recognizing specific types of relationships (e.g., proportional, linear, exponential, etc.) is a continual conceptual focus throughout this course. Students soon become far less willing to assume a relationship between two changing quantities without justification.

The students now try out their prediction processes on the three sample footprints, making their predictions for the heights based on measuring each of the footprints provided and applying their group's approach. The instructor creates a large table on the board and each group writes up their predictions for each of the sample footprints. The instructor then finally reveals the actual height of the three people whose footprints were used. Some of the groups' predicted heights are quite good for one of the footprints but none are close for the other two. The class then discusses why this might be the case and after some discussion suggestions arise around the data samples that each group collected—the size of the sample, whether the sample was chosen from an appropriate population of individuals, and of course the reasonableness of their prediction processes in the first place. The instructor at this point shares that the three footprints belonged to a child, a 28 year old woman, and the other to a 20 year old man with large feet for his height. Given the discussion about the sample population used, students can now see some reasons for the large error in two of their predictions. The class concludes with a review of all of the steps (and with learning some associated modeling language—e.g., "data," "sample," and "model") they went through to try to answer this question: (1) thinking about the question, (2) collecting data to help them answer the question, (3) analyzing that data in a variety of ways, (4) coming up with a model for the situation, (5) using that model to make a prediction, and finally (6) evaluating the accuracy of their model and the limitations of their model. The students are sent away to write up their complete solution and hand it in during the next class.

As this is the first day of class, the activity is not intended to jump immediately into the concepts of the course but instead to ask the students a "good question" that engages them from the very start of the course in the types of reasoning they'll be doing on a daily basis. This "good question" is open, unfamiliar, and unguided so as to force students to reach for their own understanding of the situation, to use that understanding to make sense of the question, and to develop an approach to answering the question. This question is set in an applied and familiar context as many of our questions are. This allows students to feel comfortable conjecturing and experimenting based upon their own prior knowledge about the situation being discussed in the question. They feel that they at least understand the situation and can picture it even if they are not sure what to do with the question yet. The applied context also allows them to reach for data to help them answer the question, a key idea that we will return to time and time again in this course. This question highlights for students that they themselves have a responsibility for evaluating how good their answer is: Have they taken into consideration the right population and the right assumptions? Did they have enough data, and did they have evidence to assume the type of relationship between the

variables that they used? Most importantly though, this question allows for multiple approaches and multiple answers—all of which could be correct. As a result, students must articulate and explain their reasoning to the class so that everyone can learn from their approach. This articulation, in turn, is an extremely important step in fully developing their own understanding. As instructors, we take the time to highlight a correct aspect of every group's approach even if they have not developed their idea fully and praise the group for this work. This can be very powerful for students who have always considered themselves terrible at math. The startling realization that many students come to on the first day is that there is more than one right answer and that theirs was one of the right answers!

For this day to set the stage for developing conceptual understanding, we must not only ask a good question but also introduce our students to the collaborative classroom environment that we will use for the rest of the semester. On the first day, students learn that when they read a question, they will not be expected to "know the method" but instead to identify what they know and what confuses them about the question. They realize that this is just the first step towards answering the question. After this initial examination, they will have to be prepared to discuss their reasoning with a few classmates and resolve some of their confusion during this small group discussion. They will also be expected to share their small group discussion with the whole class. They learn that the role of the instructor during whole class discussion will not be to tell students that they are right or wrong but instead to help the class to identify a number of key ideas from the small group work that the whole class should pursue. The students know that following class discussion, they will often return to work further on the question in their small groups or individually where they will be expected to try out some of the ideas that they or their classmates had. Finally, they learn that they must rely on their own thinking because the instructor is not going to suddenly show them "the method" for the question they are working on.

Throughout the rest of the course, some class activities allow students to build their own understanding of a concept by reasoning from prior knowledge (including the work that they have already done in this course). Other class activities ask students to expand on that understanding to adapt, extend, and apply that concept in novel situations. Most concepts in our precalculus course are encountered by our students through this same development process, simultaneously developing in our students both a deep understanding of the meaning of each concept and the ability and confidence to engage in the challenging process of independently reasoning directly from their own understanding to answer questions.

We help our precalculus students build **conceptual understanding** by actively engaging them on a daily basis in the process of having to reason from concepts (and prior knowledge) to answer questions and solve problems, both within applications and within purely mathematical contexts. How?

1. <u>Questioning</u> that forces students to actively engage in reasoning from concepts:
    - o Open, unfamiliar, unguided questions.
    - o Questions that require students to articulate and explain their reasoning.
    - o Students work on questions (individually or in small groups) prior to establishing consensus as a class - both when developing an initial understanding of the concept and when expanding/applying that understanding in synthesis.

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2. <u>Learning Environment</u> that supports and promotes the development of reasoning from concepts:
- Collaboration: In pairs, small groups, together as a class, and together with the instructor.

Expectations and Trust: Students know what is expected of them, they know nobody (including the instructor) is going to just give them the answer or provide example solutions, and they trust that the class will eventually come together with the Instructor to clarify points of confusion.

Assessing Conceptual Understanding

As instructors, we focus a huge amount of energy on providing the course activities and classroom environment that we believe will help our students develop conceptual understanding. However, for us truly to know the effectiveness of this work, we must be equally invested in the assessment of this conceptual understanding. We quickly discover that it is not easy to assess conceptual understanding, as students could correctly answer many questions intended to test a concept by using a method that they have developed or memorized while working on a similar question in previous course work. Therefore, to assess conceptual understanding, we must examine a concept in a setting that is novel enough to require that the student is indeed reasoning from an understanding of the concepts in order to correctly answer the questions.

This is easier said than done. Many students enter our class with the opinion that being given a problem on an exam which they have not seen before is not fair. In a traditional classroom, they might even be justified, as they have probably not had an opportunity to develop the ability to reason from a concept to solve an unfamiliar problem. In order to prepare students for these novel problems, we must first carefully build both their confidence to attack problems that are not of a "type" they recognize and their ability to use their understanding of concepts to answer these questions. We do this by engaging students in this type of reasoning in every class activity and in every homework set. We also discuss regularly with our students how important it will be in their future careers to be able to approach an unfamiliar problem and form a strategy for answering it. After this work in class and on homework, we can ask a question on an exam that involves a novel context or use of a concept *without the reasoning experience itself being novel or unexpected*.

Students are now prepared to see novel problems on homework and exams, leaving us with the task of writing these problems! After identifying the concepts we want to test, we attempt to write a novel question for the exam. For instance, consider the concept of characterizing and distinguishing linear and exponential growth/decay. In a course activity from this unit, students applied their understanding of this concept in a problem involving sample data. Reasoning from their understanding of the characteristics of linear and exponential growth/decay, students developed on their own a strategy for determining whether a given data set would be well represented by a linear function, an exponential function, or neither. In addition, students had to articulate why the approach that they used allowed them to draw these conclusions.

When faced with writing an exam question that tests this same concept (by applying it in a novel situation), one option is to ask the same type of question (determining whether a given data set is well represented by a linear function, exponential function,

or neither) but using different data—in a new applied setting. Even with new data in a new applied setting, however, it is possible for a student to answer this question by simply repeating the same sequence of steps (the method) that they developed and performed in the course activity without ever considering why they chose that sequence of steps the first time. This means that the student is no longer using (and thus, demonstrating) their ability to reason from an understanding of the concepts involved. Most likely, they are instead recognizing that this problem is of a very similar format to the problem on the course activity and then adapting their method from that problem to answer this one. As a result, a problem that might have assessed conceptual understanding very well in another precalculus classroom becomes more of a method problem in our classroom *because of our students' previous experience with that concept in the context of our course*.

So, we look for additional changes to the data that would render the method developed in the course activity no longer applicable in the exam problem. One possibility is changing the data from being decreasing (like in the course activity) to being increasing data in the exam question. This is not sufficient, though, because the same sequence of steps still easily applies. However, changing the data so that there are *not* consistent increments between data values (as there were in the course activity) does force the students to alter their strategy significantly for answering the question. The student would then have to return to their understanding of the mathematical concept in order to answer the question. Thus, conceptual understanding is being assessed by this question—to a certain extent.

It is possible to ask the student to demonstrate an even deeper level of conceptual understanding. This can be achieved by having the student either interpret the meaning of the mathematical characteristic in the context of the new applied setting or make connections between different representations (numerical, graphical, symbolic, or narrative) of the mathematical characteristic. In *Understanding by Design*, Wiggins and McTighe (2001) have developed a "multifaceted view of what makes up a mature understanding, a six-sided view of the concept." The six facets that they mention are "can explain, can interpret, can apply, have perspective, can empathize and have self-knowledge." In the questions that require this deeper conceptual understanding, our Precalculus students are demonstrating the first four of these facets.

While considering how well the exam questions that we had written assessed our students' conceptual understanding, we realized that we were asking ourselves a series of questions about each problem. We decided to try to formalize this questioning process so that we would then have a tool for evaluating the extent to which conceptual understanding is being assessed by a given problem in our course. That tool could then be used to weight the scoring of exam problems, giving us scores for our students (separate from their exam scores) that indicate more specifically their level of conceptual understanding on that exam. In addition, this tool also gives us the ability to track our students' proficiency at specific types of mathematical understanding.

To begin with, we recognized that in every math course, there are computational and/or algebraic skills needed for the work in that course. We want our students to become proficient with these skills, so we put some problems on exams that assess these skills. These skills problems, however, do not require any conceptual

understanding so our weighting system should assign them a weight of zero. We then began to realize that on some of the problems that we had specifically written to test students' conceptual understanding, students were in fact more likely answering the problem by adapting methods that they had developed previously in class work (as seen in Problem 4 above). Being able to adapt a method in this way demonstrates some conceptual understanding but not as much as when a student must reason directly from the meaning of a concept. Our weighting system should assign these types of problems a weight of one. If no previously developed method can be applied to solve the problem, we assume that students must be reasoning from concepts in order to answer that question. Our weighting system should assign a weight of at least two to these types of problems. If *in addition* students were required to interpret the meaning of a mathematical characteristic in a novel applied setting, and/or the students were required to make connections between different representations (numerical, graphical, symbolic or narrative) of a mathematical characteristic, then students are demonstrating an even deeper understanding of concepts when successfully answering that question. To reflect this, these types of problems should receive a weight of three. This Conceptual Understanding Weighting System (CUWS) weights each problem (weights ranging from 0 to 3) according to the extent to which the problem assesses students' conceptual understanding, depending upon both the characteristics of the problem itself and how a student has previously encountered (in this course) the concept/s being tested by the problem.

Conceptual Understanding Weighting System (CUWS)

Check 1 (Skills): Does this problem involve only computational/algebraic skills or memorized facts with no understanding of the concept required? Yes - assign a weight of 0 to the problem and skip the remaining checks; No - continue to the next check.

Check 2 (Method): Could this problem be answered completely using a method that the student or teacher might have developed in prior course work? Yes - assign a weight of 1 to the problem and skip the remaining checks; No - continue to the next check.

Check 3 (Conceptual Reasoning Characteristics): Does the problem involve either of the following? No - assign a weight of 2 to the problem; Yes - assign a weight of 3 to the problem.
- interpreting the meaning of a mathematical characteristic in a novel applied setting
- making connections between different representations (numerical, graphical, symbolic or narrative) of a mathematical characteristic

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Results of Assessment

In order to evaluate how successful we had been in helping our precalculus students build conceptual understanding, we assessed our students' conceptual understanding in two main ways: (1) semester-long, course embedded assessment (using our course exams), and (2) pre-post assessment of standard precalculus concepts (using the Precalculus Concept Assessment (PCA)). Student scores (on course exams and the PCA) reflected different types of mathematical understanding. We applied the

Conceptual Understanding Weighting System described above to our course exams and to the PCA results in order to organize our assessment data and extract information about our students' proficiency in specific categories of mathematical understanding. For each assessment, we calculated both *conceptual scores (weighted*) for each student and levels of class proficiency in specific categories of mathematical understanding.

The tables below give sample (combined) results for the authors' Precalculus Fall 2009 sections. For each exam question the points out of 100 are provided in the second column and the weights resulting from applying the CUWS to that question are provided in the third column. The final column provides the average student score (in percent) for that question.

Table 1: Exam 1 Results

Question	Points	CUWS Weight	Average Percent Score on Each Question (58 students)
1	10	1	83.79
2	10	1	84.14
3	20	1	79.14
4	12	1	76.01
5	8	1	67.67
6	15	1	81.84
7	15	3	72.52
8	10	3	66.72

For each student, we can consider both a typical raw score, a weighted score, and proficiency levels within the different weighting categories. For example, consider a student who got a raw score of 88% on Exam 1 whereas their weighted score on the same exam was 84%. This tells us that this student has not done as well on the questions assigned a weight of 3 as they did on the questions assigned a weight of 1. This is also demonstrated by calculating a proficiency level for this student within each weighting category. For questions assigned a weight of 1 the student scored 92% whereas for questions assigned a weight of 3 the student scored 76%.

The mean raw student score on Exam 1 (without any weighting) was 76.98%. The mean weighted score when using the CUWS weightings provided in the table above was 74.72%. After applying the CUWS to Exam 1 it is evident that it mainly assessed a student's ability to adapt a previously learned method (reflected in the weighting above where six questions were assigned a weight of 1 whereas only two questions were assigned a weight of 3). Although adapting a previously learned method is something we want our students to be able to do, it is an intellectual task requiring only minimal conceptual understanding as discussed above. We would prefer that our exams instead have types of questions from each of the weighting categories mentioned above: skills

questions, method questions, conceptual questions, and questions that are assessing deep conceptual understanding. However, the difference between the weighted score (74.72%) and the raw mean (76.98%) still demonstrates that students are finding the questions weighted 3 more difficult. This is also reflected when we consider the overall mean proficiency score within each weighting category: mean on questions of weight 1 = 79.24%, mean on questions of weight 3 = 70.21%. Although we consider 70% to be a very good mean score on deep conceptual questions over a diverse group of students we hesitate to draw any conclusions from the results of this exam due to the fact that there are only 25% of the questions (and points) assigned a weight of 3.

Table 2: Exam 2 Results

Question	Points	CUWS Weight	Average Percent Score on Each Question (54 students)
1	10	1	84.44
2	9	3	82.92
3	5	1	82.96
4	20	3	47.04
5	13	1	62.68
6	10	1	83.70
7	8	3	60.88
8	15	3	50
9	10	1	73.33

The mean raw student score on Exam 2 (without any weighting) was 65.88%. The mean weighted score when using the CUWS weightings provided in the table above was 60.87%. This points to the fact that our students are doing better on the questions weighted 1 (adapting a method) than the questions weighted 3 (requiring deep conceptual understanding). This is again supported when we consider the proficiency level in each of these weighting categories. 65% of students scored 70% or higher on the questions weighted 1 whereas only 33% of students scored 70% or higher on questions weighted 3. This reaffirms our belief that students are more proficient at adapting a method that they have already used than answering a truly novel question and shows that on this exam 33% of our students demonstrated deep conceptual understanding.

Table 3: Precalculus Concept Assessment (PCA) Results

Problem	Points	CUWS Weight	Number of Students with this Question Correct (Out of 50 students)
1	4	2	34
2	4	3	11
5	4	2	11
6	4	2	12
7	4	3	15
8	4	3	3
9	4	1	21
10	4	3	10
11	4	1	9
15	4	3	9
18	4	3	32
19	4	1	22
20	4	3	20
21	4	1	3
22	4	3	10

The Precalculus Concept Assessment (used with the permission of Marilyn Carlson (2009)) is an assessment that the students *do not study for*, that does not factor into their course grade, and that poses a high level of difficulty for most of our students. With this lack of preparation or incentive, it was not surprising that most of our students did not score very well on this assessment. However, one interesting (and unexpected) result did emerge when proficiency levels within each weighting category were compared: On questions assigned a weighting of 1, the mean student score was 27.5%; on questions assigned a weighting of 2, the mean student score was 38%, and on questions assigned a weighting of 3, the mean student score was again 27.5%. This is very different from the results seen in course exams (Exam 1 and Exam 2) where students were performing better by far on the questions weighted 1. It makes sense that on a test that students have not studied for, they are not drawing on previously developed methods to answer the questions but instead are more likely reasoning from concepts. It is also interesting that here (where for the first time we see the distinction between questions weighted 2 and questions weighted 3) students are doing significantly better on the questions weighted 2 than on the questions weighted 3.

Conclusions

Over the course of writing this article, we have discovered a great deal about both teaching and student learning in our classrooms. We have come upon these discoveries in a variety of ways: some from the act of articulating what exactly it is we do in course

activities to allow students to develop their understanding of a concept, some from examining exactly how we assess (or have failed to assess in some cases) that conceptual understanding, and still more from analyzing the data that we collected over the course of the semester under study.

- We are happy with our current approach to developing conceptual understanding as a whole. However, based on the analysis of exam questions and the PCA, we have already planned improvements to many of our course activities. In addition, we plan to create some new course activities to address concepts that we had not previously discussed in this course. We will also map each exam and PCA question to the course activity/activities that developed the concepts being assessed in that question. With this mapping we can then use student results on the exams and the PCA more effectively to identify the course activities that are not building conceptual understanding to the extent that we would like and revise those course activities accordingly.

- Our assessment of conceptual understanding is still a work in progress. The process of writing the paper has opened our eyes to the limitations of our current exams. Using our new CUWS to help us evaluate the questions as we write them should enable us to improve our assessment of conceptual understanding not just in Precalculus but also in our other math courses.

- Our development of the CUWS itself is only in the preliminary stages. We have developed a student Survey of Mathematical Thinking (SoMT) that is administered to students after the exam. This asks students to describe how they approached some of the questions on the exam. We can in turn use this to compare to our weighting with the CUWS to see if students are approaching questions in the way that we thought. We have administered this survey once and are analyzing the data currently.

- All of our work to develop and assess conceptual understanding is part of a larger project working on these same questions in many courses that we teach. We have developed a student centered, active-learning curriculum for Calculus I and II using the same ideas discussed in this paper and plan to use the CUWS to help in assessing conceptual understanding in these courses as well.

- In the course of our research, we are more effectively engaging our students as partners in the learning process. As we ourselves have come to better understand the subtleties of building and assessing conceptual understanding, we have been able to communicate more clearly to our students why we do the things we do in our course and what their role is in the learning experience.

As with much SOTL work, we did not begin this work with the goal of developing an SOTL project. For many years in our department, there has been a firm belief that actively engaging students during class time improves conceptual understanding. Recently, however, we began to notice that although students were active during class time they were sometimes not developing the level of conceptual understanding that we expected. We reflected more carefully on what exactly students were actively engaged in during class and we decided that in many cases, they were still not being asked to reason from concepts to develop their own understanding. We realized that we were

guiding the students so much in the course activities that we were still the ones engaging in this reasoning process and the students were simply following our plan while missing the big picture. We decided to revise each of our activities to try to incorporate a more open approach that would allow students to develop their own understanding of a concept. We were very happy with the results and how these new activities worked in our classroom but at the same time we were unable to be certain that students were, indeed, developing deep conceptual understanding.

This was probably the first time that we began to think of this project as an SOTL project. We wanted to investigate whether the understanding that we believed was being developed truly was. We developed and implemented a plan to collect data to investigate our claim using an external concept exam as well as course embedded assessment. When analyzing this data we realized another difficulty: How could we identify which questions were assessing a student's ability to reason from concepts as opposed to assessing our students proficiency with skills or ability to adapt a mathematical method? We studied our exam questions carefully and found that many of the questions we had written to assess conceptual understanding were not in fact as good as we had hoped. This was disappointing at first but it led us to the interesting problem of how to identify and write problems which require students to reason from concepts.

In doing this, we developed the Conceptual Understanding Weighting System, which not only helps us to focus in on the evidence of conceptual understanding in our current data but also we believe will help us to write better exam problems in the future. The most surprising consequence of our work on this project was the direct impact approaching our questions from an SOTL perspective had on our classroom experience. Developing our research questions, designing our investigation plan, and then collecting, analyzing, and drawing conclusions from data directly impacted how we now write activities, exams, and homework, thus having a direct impact on our students' learning experience.

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